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AERONAUTICAL RESEARCH LABORATORIES

MELBOURNE, VICTORIA

Systems Report 7

CONFIDENCE LIMITS FOR SYSTEM RELIABILITY

P. F. PRESTON

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SYSTEMS REPORT 7

CONFIDENCE LIMITS FOR SYSTEM RELIABILITY

by

P. F. PRESTON

SUMMARY

The statistical problem of constructing confidence limits for system reliability from pass-fail data on the components is discussed. Usually such data are obtained from fixed sample size tests on the components. Exact and approximate methods for dealing with series systems are reviewed. A new approximate method is proposed and its performance evaluated in a number of special cases.

An alternative approach that avoids some of the problems inherent in the above methods is to use variable sample size testing of the components. A sequential experimental design that is applicable to any coherent structure is given.

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ABSTRACT

7. *This Report is concerned with the statistical problem of constructing confidence limits for system reliability from pass-fail data on the components. A new approximate method for dealing with series systems is proposed.*

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CONTENTS

1. INTRODUCTION

Consider a system with m components. Let y_i $(i = 1, 2, \ldots, m)$ be an indicator of the state of component i (i.e. $y_i = 1$ if component i is good and $y_i = 0$ if component i fails). Similarly let y be an indicator of the state of the system. The boolean transmission function

$$y = f(y_1, y_2, \ldots, y_m)$$

relates the performance of the system to the performance of its components, and is called the "structure function".

The systems considered in this note are coherent systems whose components operate independently and whose structure function is known.

The reliability of the system is

$$p = Pr\{y = 1\}.$$

Some examples of types of systems are:

(i) *Series system*

A series system is good only if all its components are good. It has structure function

$$f(y_1, y_2, \ldots, y_m) = \prod_{i=1}^{m} y_i.$$

(ii) *Parallel system*

A parallel system is good if any one of its components are good. It has structure function

$$f(y_1, y_2, \ldots, y_m) = 1 - \prod_{i=1}^{m} (1 - y_i).$$

(iii) "*k out of m*" *system*

A "k out of m" system is good if any k of its m components are good. The structure function cannot be expressed in simple form.

In practice, systems are often composed of blocks where the components within a block are linked together by an arrangement such as one of those given above, and the blocks themselves are linked together by some other arrangement. Lloyd and Lipow[1] (p. 255) cite the case of a space vehicle's temperature control system which consists of a number of "k out of m" subsystems linked together in a series—parallel arrangement.

It is often too expensive to test the system a number of times to determine its reliability. What can be done however, is to test the components individually. This note is concerned with the problem of making inferences about system reliability on the basis of pass-fail data obtained from tests on the components. Inferences generally fall into two classes, point estimates and confidence limits. As pointed out by Easterling[2], the construction of point estimates is relatively straightforward. If the results for component i are x_i successes in n_i tests $(i = 1, 2, \ldots, m)$ an estimate of the reliability of component i is

$$\hat{p}_i = x_i/n_i$$

and the maximum likelihood estimate of the system reliability p is

$$\hat{p} = h(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)$$

where $h(p_1, p_2, \ldots, p_m)$ is a function giving system reliability in terms of the component reliabilities.

Construction of confidence limits is by no means so straightforward. The case that has received the most attention is the case of a series system. Exact methods have been developed,

notably by Buehler[3], but these are generally too cumbersome computationally, especially when the number of components is greater than two and the sample sizes (n_i) are not all equal. A number of approximate methods have been proposed to overcome this difficulty. In our opinion the best of these is the one due to Easterling[2]. A new approximate method is proposed in this note and its performance evaluated in certain special cases. The indications are that this method is comparable in overall performance to the method of Easterling[2], and has advantages in the case when no failures are observed on the component with the smallest sample size.

The remarks above refer to fixed sample size testing of the components, i.e. cases where the sample sizes are fixed in advance and do not depend on the results obtained in the tests. An alternative approach was suggested by Winterbottom and Verrall[4]. They gave a sequential testing method, applicable to series or parallel systems, that minimizes the number of component tests needed and produces results in a form that enables easy construction of confidence limits. This method was generalized by Preston[5] to deal with any system whose structure function is known.

2. "EXACT" BINOMIAL CONFIDENCE LIMITS

In this section the theory of "exact" binomial confidence limits is outlined; for a fuller development the reader is referred to Mood and Graybill[6] or Kendall and Stuart[7].

The binomial probability function with sample size n and probability p is denoted by $B(n, p)$ and is given by

$$Pr\{X = x\} = \binom{n}{x} p^x (1 - p)^{n-x}.$$

A lower confidence limit for p at a level of confidence α based on an observation x is denoted by $L_\alpha(x)$. The conditions that a set of numbers $L_\alpha(x)$ $(x = 0, 1, \ldots, n)$ must obey in order to constitute a reasonable set of lower confidence limits are:

$$Pr\{L_\alpha(X) \leqslant p\} \geqslant \alpha \quad \text{for all } p \text{ in } (0, 1), \tag{1}$$

$$\text{if } x_1 < x_2 \quad L_\alpha(x_1) \leqslant L_\alpha(x_2), \tag{2}$$

$$\text{and } L_\alpha(x) \text{ should be as large as possible} \tag{3}.$$

Condition (1) guarantees that the limits do achieve the intended confidence; condition (2) is called the regularity condition and ensures that the greater the observed value of x, the greater the confidence limit; and condition (3) states that, subject to conditions (1) and (2), the limits should be as large as possible (after all $L_\alpha(x) = 0$ for all x obeys conditions (1) and (2)).

The limits generated by these conditions are given by the solutions of the equation

$$\sum_{j=x}^{n} \binom{n}{j} [L_\alpha(x)]^j [1 - L_\alpha(x)]^{n-j} = 1 - \alpha \tag{4}$$

$$\text{for } x = 1, 2, \ldots, n$$

where $L_\alpha(o)$ is defined to equal 0. An alternative to solving equation (1) for $L_\alpha(x)$ is to use the identity

$$\sum_{j=x}^{n} \binom{n}{j} [L_\alpha(x)]^j [1 - L_\alpha(x)]^{n-j} = I(L_\alpha(x), x, n - x + 1) \tag{5}$$

where $I(s, \alpha, \beta)$ is the incomplete beta function

$$I(s, \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^s t^{\alpha-1}(1 - t)^{\beta-1} dt$$

which is tabulated in Pearson[8].

The limits generated by (4) are referred to as "exact" because they obey condition (1). The proof of this proposition is given by Lloyd and Lipow[1] p. 209. They are exact in the sense that they actually attain the confidence α, in contrast to such approximate limits as those based on the normal approximation to the binomial (see Hogg and Craig[9] p. 196) which are not guaranteed to do so. They are not exact in the sense, used by some authors, that they obey

$$Pr\{L_\alpha(X) \leqslant p\} = \alpha \tag{6}$$

for all p. It is only possible to construct a set of intervals that are exact in this sense by a post-experiment randomization procedure (see Lloyd and Lipow p. 212) but this procedure is not widely used in practice.

Looking into this point a little further, Table 1 shows the values of $L_\alpha(x)$ for $x = 0, 1, \ldots, n$ when $n = 10$ and $\alpha = .9$, and Figure 1 shows the achieved confidence level $\alpha^*(p)$ where

$$\alpha^*(p) = Pr\{L_\alpha(X) \leqslant p\},$$

as a function of p. It can be seen that $\alpha^*(p)$ is equal to α for those values of p for which

$$L_\alpha(x) = p, \quad x = 1, 2, \ldots, 10$$

and is greater than α for all other values of p. It can also be seen that $L_\alpha(x)$ constitute the greatest set of numbers that will achieve confidence α; if any one of them was increased the confidence would go below α for some p. For instance, if $L_{.9}(9)$ was increased from 0·6632 to 0·68 the achieved confidence level ($\alpha^*(p)$) would be lower than 0·9 for p in the interval (0·6632, 0·68) (see the dotted curve in Figure 1).

It is unfortunate that the achieved level is so much higher than the intended level for most values of p. This phenomenon is a consequence of the fact that the binomial is a discrete distribution. It does not occur with continuous distributions. In the next section some approximate methods for finding confidence limits from binomial data on the components of a series system will be discussed. One criterion for evaluating these methods that will be used is the comparison of the intended and achieved confidence levels. To set the scene for this sort of comparison, in the remainder of this section we will look at the confidence level achieved by one of the commonly used approximate methods for constructing binomial confidence limits (see Hogg and Craig[9] p. 196).

The binomial distribution $B(n, p)$ has mean np and variance npq. The binomial can be approximated by the normal distribution by equating the mean and variance. Thus $\dfrac{X - np}{\sqrt{npq}}$ is approximately normal with mean 0 and variance 1. An estimator of the variance is $n\left(\frac{X}{n}\right)\left(1 - \frac{X}{n}\right)$, and it can be shown that $\dfrac{X - np}{\sqrt{n\frac{X}{n}\left(1 - \frac{X}{n}\right)}}$ is asymptotically normal with mean 0 and variance 1. Then, from tables of the normal distribution,

$$Pr\left\{\frac{X - np}{\sqrt{X\left(1 - \frac{X}{n}\right)}} \leqslant 1{\cdot}282\right\} = 0{\cdot}9$$

That is $Pr\left\{\left(X - 1{\cdot}282\sqrt{X\left(1 - \frac{X}{n}\right)}\right)n^{-1} \leqslant p\right\} = 0{\cdot}9$

Thus $$A_{.9}(x) = \left(x - 1{\cdot}282\sqrt{x\left(1 - \frac{x}{n}\right)}\right)n^{-1}$$

is an approximate 0·9 confidence limit for p. Table 1 shows the exact ($L_{.9}(x)$) and approximate

$(A.9(x))$ limits when $n = 10$, and Figure 2 shows the confidence levels achieved by these approximate limits.

It should be noted that this demonstration is not intended to denigrate the method of constructing confidence limits based on the normal approximation. The theory on which it is based is asymptotic, and it is well known that it should only be used when n is reasonably large, say 30 (see Crow, Davis and Maxfield[10] p. 51). The demonstration is intended to be a simple example of the sort of comparisons we shall be making in later sections.

3. EXACT CONFIDENCE LIMITS FOR SYSTEM RELIABILITY

Consider a system with m components linked in series. If the reliability (probability of successful operation) of the ith component is p_i, the system reliability p is given by

$$p = \prod_{i=1}^{m} p_i.$$

Suppose further that binomial testing has been carried out on each component giving x_i successes from n_i tests on component i $(i = 1, 2, \ldots, m)$. The problem we address is the determination of a lower confidence limit for p on the basis of these data. The theory of exact intervals as given by Buehler[3] will be discussed in this section, and some of the better approximate methods will be reviewed in the next section.

A lower confidence limit for p based on the data $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, denoted by $L_\alpha(\mathbf{x})$, must obey a similar set of conditions to those given in the previous section for the case where only one item has been tested. Certainly it should obey

$$Pr\{L_\alpha(\mathbf{x}) \leqslant p\} \geqslant \alpha \tag{7}$$

and also $L_\alpha(\mathbf{x})$ should be as large as possible, (8)

however, the appropriate regularity condition poses something of a problem. For one component the regularity condition is straightforward, it amounts to ordering the possible outcomes of the experiment in terms of extremeness. Certainly the result $x = n$ (n successes in n tests) is more extreme than the result $x = n - 1$, and $x = n - 1$ is more extreme than $x = n - 2$, and so on; thus one requires $L_\alpha(n)$ to be greater than $L_\alpha(n - 1)$, and $L_\alpha(n - 1)$ to be greater than $L_\alpha(n - 2)$. But with more than one component it is sometimes difficult to decide if one result is more extreme than another. For instance, with two components each tested 10 times, is the result $x_1 = 9$, $x_2 = 7$ more or less extreme than the result $x_1 = 8$, $x_2 = 8$?

The first step in constructing a set of limits $L_\alpha(\mathbf{x})$ for all $\mathbf{x}$ is to define an order for the vectors $\mathbf{x}$. Let us suppose for the moment that this has been done (three criteria for ordering that have been suggested will be discussed later) and denote the order by $\mathbf{x}_j$, $j = 1, 2, \ldots, N$, where

$$N = \prod_{i=1}^{m} (n_i + 1).$$

The regularity condition implied by this order is

$$L_\alpha(\mathbf{x}_j) \leqslant L_\alpha(\mathbf{x}_k) \text{ if } j < k. \tag{9}$$

Given the conditions (7), (8) and (9) the set of largest limits are given by

$$L_\alpha(\mathbf{x}_j) = \inf\{p \mid \sum_{k=j}^{N} B_k = 1 - \alpha\} \tag{10}$$

where $B_k = \prod_{i=1}^{m} \binom{n_i}{x_{ik}} p_i^{x_{ik}} (1 - p_i)^{n_i - x_{ik}}$

and $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$.

It can be seen that, in the case of one component ($m = 1$) equation (10) gives the same solution for $L_\alpha(x)$ as equation (4), by noting that the term

$$\sum_{k=j}^{N} B_k$$

reduces exactly to the lefthand side of equation (4). For a fuller explanation of the derivation of equation (10) the reader is referred to Buehler[3] and Winterbottom[11].

There are two difficulties that arise when using this exact method in practice, one is defining the order $\mathbf{x}_j$, and the other is solving equation (10). The latter problem can be illustrated with an example. Suppose $m = 2$, $n_1 = 10$, $n_2 = 7$, $N = 88$, and the first part of the order has been defined to be:

$$\begin{aligned}
\mathbf{x}_{88} &= (10, 7)\\
\mathbf{x}_{87} &= (9, 7)\\
\mathbf{x}_{86} &= (10, 6)\\
\mathbf{x}_{85} &= (8, 7)\\
\mathbf{x}_{84} &= (9, 6)\\
\mathbf{x}_{83} &= (7, 7)\\
\mathbf{x}_{82} &= (8, 6),
\end{aligned}$$

and that we wish to find $L_\alpha(\mathbf{x}_{84})$, i.e. the lower confidence limit for p when 9 successes out of 10 tests were obtained on component 1 and 6 successes out of 7 tests were obtained on component 2. Equation (10) gives

$$L_\alpha(\mathbf{x}_{84}) = \inf\{p \mid \sum_{k=84}^{88} B_k = 1 - \alpha\}$$

where $p = p_1 p_2$

and
$$B_{84} = \binom{10}{9} p_1^{\,9}(1 - p_1)\binom{7}{6} p_2^{\,6}(1 - p_2)$$

$$B_{85} = \binom{10}{8} p_1^{\,8}(1 - p_1)^2 p_2^{\,7}$$

$$B_{86} = p_1^{\,10}\binom{7}{6} p_2^{\,6}(1 - p_2)$$

$$B_{87} = \binom{10}{9} p_1^{\,9}(1 - p_1) p_2^{\,7}$$

$$B_{88} = p_1^{\,10} p_2^{\,7}.$$

Now $\sum_{k=84}^{88} B_k$ can be written

$$\sum_{k=84}^{88} B_k = \sum_{i=9}^{10} \sum_{j=6}^{7} \binom{10}{i} p_1^{\,i}(1 - p_1)^{10-i}\binom{7}{j} p_2^{\,j}(1 - p_2)^{7-j} + B_{85}. \tag{11}$$

The point to be noted here is that in general $\sum_{k=j}^{N} B_k$ *cannot* be written as a neat sum of the form

$$\sum_{i_1 = x_{1j}}^{n_1} \sum_{i_2 = x_{2j}}^{n_2} \cdots \sum_{i_m = x_{mj}}^{n_m} \left[\binom{n_1}{i_1} p_1^{\,i_1}(1 - p_1)^{n_1 - i_1} \cdots \binom{n_m}{i_m} p_m^{\,i_m}(1 - p_m)^{n_m - i_m}\right]$$

but in general is a rather messy collection of binomial product terms (as is shown by the fact that B_{85} cannot be included in the summation on the righthand side of equation (11)). This makes the minimization procedure needed to find

$$\inf\{p \mid \sum_{k=j}^{N} B_k = 1 - \alpha\}$$

a complicated computational exercise, and in fact effectively rules out its use in practice for cases when m is greater than 3.

Turning now to the problem of ordering the vectors $\mathbf{x}$, three methods have been suggested. Buehler[3] noted that if $L_{\alpha_i}(x_i)$ is an exact α_i binomial lower confidence limit for p_i then

$$\prod_{i=1}^{m} L_{\alpha i}(x_i) \tag{12}$$

is a lower limit for p with confidence at least $\prod_{i=1}^{m} \alpha_i$. This limit in itself is not useful because it is generally considerably smaller than it needs to be to achieve the confidence $\prod_{i=1}^{m} \alpha_i$. However, if one takes

$$\alpha_i = \alpha^{1/m} \qquad i = 1, 2, \ldots, m$$

one can order the $\mathbf{x}$'s in such a way that the limits given by (12) form an increasing set. This order, suggested by Buehler[3], was presumably the one used by Lipow and Riley[12] in constructing their tables using the exact method given by equation (10). The tables of Lipow and Riley provide lower limits on the reliability of systems composed of one, two or three components when the same number of tests has been carried out on each component.

The second method for ordering is based on the uniformly minimum variance estimate of p and will be referred to as the "UMVE" method (see Winterbottom[11]). The UMVE estimate of p is

$$\hat{p} = \prod_{i=1}^{m} x_i/n_i.$$

The vectors $\mathbf{x}$ can be ordered so that, if $j < k$

$$\prod_{i=1}^{m} x_{ij}/n_i \leqslant \prod_{i=1}^{m} x_{ik}/n_i .$$

This order has intuitive appeal in that if a set of data $\mathbf{x}_j$ give rise to a larger estimate of p than another set $\mathbf{x}_k$ one would also expect it to give a larger lower confidence limit.

The third method for ordering is given by Winterbottom[11] and is called the largest possible "LP" method. Here

$$\mathbf{x}_N = (n_1, n_2, \ldots, n_m)$$

and, given $\mathbf{x}_N$, $\mathbf{x}_{N-1}$ is the vector $\mathbf{x}$ that gives the largest $L_\alpha(\mathbf{x}_{N-1})$, and given $\mathbf{x}_N$ and $\mathbf{x}_{N-1}$, $\mathbf{x}_{N-2}$ is the vector giving the largest $L_\alpha(\mathbf{x}_{N-2})$, and so on. This method appears to give a set of limits that are uniformly large, but it involves even more computation than the other methods.

4. APPROXIMATE CONFIDENCE LIMITS FOR SYSTEM RELIABILITY

It can be seen from the comments made above that exact methods are usually too complex to be used in practice, and so approximate methods must be resorted to. Some of the more promising approximate methods that have been proposed are discussed in this section.

The first approximate method is attributed to Lindstrom and Madden and is given in Lloyd and Lipow[1] p. 226. It will be referred to as the LM method. The method is very simple and runs as follows. Given the results x_i successes from n_i trials on component i, $i = 1, 2, \ldots, m$ one first estimates p by

$$\hat{p} = \prod_{i=1}^{m} x_i/n_i .$$

The collection of n_i trials ($i = 1, 2, \ldots, m$) on the components should at least be equivalent to $n^* = \min(n_i)$ trials on the whole system. If one had carried out n^* trials on the system and had obtained $x^* = n^* p$ successes one would have got the same estimate of p as given above.

Thus the data x^* successes in n^* trials is taken as equivalent to the component data, and the LM lower confidence limit is the exact binomial limit based on x^* and n^*. If x^* is not an integer the limit is found by interpolation.

The LM method is attractive because it is simple enough to be executed by hand with the aid of a set of tables of binomial confidence limits. However, it does appear to make rather inefficient use of the data in some cases. For instance, when $n_1 = n_2 = 20$, $x_1 = 15$, $x_2 = 16$ we have

$$p = 0{\cdot}6$$
$$n^* = 20$$
$$x^* = 12$$

and the LM lower 0·9 confidence limit is 0·433 which seems reasonable; however when $n_1 = 40$, $n_2 = 20$, $x_1 = 30$, $x_2 = 16$ exactly the same results are obtained, which seems unreasonable.

The second approximate method, due to Easterling[2], is similar in concept in that it determines system results that are deemed to be equivalent to the component results. It has the advantage of apparently making better use of the data at the expense of being slightly more complicated. Easterling begins with the estimate of p given above and notes that, by the theory of maximum likelihood estimation, it has asymptotic variance

$$\sigma^2 = \sum_{j=1}^{m} \prod_{\substack{i=1 \\ i \neq j}}^{m} p_i^2 \operatorname{Var}(\hat{p}_j) \tag{13}$$

where $\hat{p}_i = x_i/n_i$

and $\operatorname{Var}(\hat{p}_i) = p_i q_i/n_i,\ i = 1, 2, \ldots, m.$

An estimate $\hat{\sigma}^2$ of σ^2 can be constructed by replacing p_i by $\hat{p}_i$ in (13).

Easterling then defines the equivalent set of system results as those values $\hat{x}$ and $\hat{n}$ (representing $\hat{x}$ successes out of $\hat{n}$ trials on the system) that give rise to the same reliability estimate $\hat{p}$ and the same variance estimate $\hat{\sigma}^2$ as the component data. System results $\hat{x}, \hat{n}$ give a reliability estimate $\hat{x}/\hat{n}$ and a variance estimate $(\hat{x}/\hat{n})(1-\hat{x}/\hat{n})/\hat{n}$. Thus $\hat{x}$ and $\hat{n}$ must obey:

$$\hat{x}/\hat{n} = \hat{p} = \prod_{i=1}^{m} x_i/n_i \tag{14}$$

and $$(\hat{x}/\hat{n})(1 - \hat{x}/\hat{n})/\hat{n} = \hat{\sigma}^2 = \sum_{j=1} \prod_{\substack{i=1 \\ i \neq j}} \hat{p}_i^2\, \hat{p}_j\, (1 - \hat{p}_j)/n_j\,. \tag{15}$$

These equations are easily solved. From (15)

$$\hat{n} = \hat{p}\,(1 - \hat{p})/\hat{\sigma}^2 \tag{16}$$

and from (14)

$$\hat{x} = \hat{n}\,\hat{p}\,. \tag{17}$$

The lower confidence limit for system reliability is then taken as the exact binomial limit obtained from $\hat{x}$ and $\hat{n}$. If $\hat{x}$ and $\hat{n}$ are not integers the limit can be found either by interpolation or by using values of $\hat{x}$ and $\hat{n}$ rounded up to the next integer. Following the notation used by both Easterling[2] and Winterbottom[11] the limits obtained by interpolation will be referred to as "MML" limits and the limits obtained by rounding up as "MMLI" limits.

As an example, with the data $n_1 = 10$, $x_1 = 8$, $n_2 = 7$, $x_2 = 6$, we have:

$$\hat{p} = 0{\cdot}686,\ \hat{p}_1 = 0{\cdot}8,\ \hat{p}_2 = 0{\cdot}8571$$

$$\hat{\sigma}^2 = \hat{p}_1^2\, \hat{p}_2\, (1 - \hat{p}_2)/n_2 + \hat{p}_2^2\, \hat{p}_1\, (1 - \hat{p}_1)/n_1$$

$$= 0{\cdot}02295$$

From (16) $\hat{n} = 9{\cdot}39$

and from (17) $\hat{x} = 6{\cdot}44.$

Then the MMLI lower 0·9 limit, based on the rounded up values of 7 successes from 10 trials, is 0·448.

It should be noted that Easterling[2] gave his theory in terms of a general system with known reliability function

$$p = h(p_1, p_2, \ldots, p_m),$$

and stated that the maximum likelihood estimate $h(\hat{\boldsymbol{p}})$ has asymptotic variance

$$\sigma^2 = \prod_{i=1}^{m} \left[\frac{\partial h(\boldsymbol{p})}{\partial p_i}\right]^2 \text{Var}\,(\hat{p}_i). \tag{18}$$

In the general case the method is approximate in the sense that the asymptotic variance is substituted for the exact variance. In the case under consideration, i.e. the case of a series system where

$$h(p_1, p_2, \ldots, p_m) = \prod_{i=1}^{m} p_i,$$

equation (18) reduces to equation (13).

There are a number of other methods based on asymptotic approximations to the distributions of certain statistics. For instance, it can be shown that the maximum likelihood estimate

$$\hat{p} = \prod_{i=1}^{m} x_i/n_i$$

is asymptotically normally distributed with mean p and variance σ^2 given by (13). This enables one to construct confidence limits in the usual way using the normal distribution. This particular method suffers from the fact that the normal distribution is symmetrical and unbounded and for small sample sizes is not a good approximation to the true distribution of $\hat{p}$ which is asymmetrical (in general) and bounded on (0, 1). The method sometimes gives confidence limits for p which lie outside the interval (0, 1). This method is known as the asymptotic maximum likelihood method (AML).

Other methods based on approximating distributions are given by Madansky[13] and Mann *et al.*[14] While these appear to be superior to the normal approximation method mentioned above the comparisons of Easterling[2], Winterbottom[11], and Mann *et al.*[14] indicate that the methods of Lindstrom and Madden, and Easterling have the most to offer in cases where the component sample sizes are reasonably small. Connor and Wells[15] proposed a method based on similar principles to the LM method but requiring much more computation. Winterbottom[11] indicates that it has no significant advantages over the LM method. The reader is referred to Mann *et al.*[14] for descriptions of these methods.

5. A NEW METHOD

In this section we propose a new approximate method for finding confidence limits for the reliability of series systems.

The method is similar to the LM and MML/MMLI methods in that, for the purpose of constructing confidence limits, the set of component results is replaced by "equivalent" system results. Thus, by a series of operations, the data $x_1 : n_1$, $x_2 : n_2, \ldots, x_m : n_m$ are progressively reduced to $x : n$, which is regarded as representing x successes from n trials on the system. The "exact" binomial theory of section 2 is then used to give the required confidence limit.

Two concepts need to be introduced in order to describe the operations involved in "reducing" the component results to the equivalent system results. The first is the idea of a "substitute" result. Given x_i successes from n_i trials on component i, represented by $x_i : n_i$, a "substitute" result is defined as a pair $a : b$, where a and b are any real numbers obeying

$$\frac{a}{b} = \frac{x_i}{n_i}$$

and $b < n_i$.

For instance, 4 : 5 *is a substitute result for* 8 : 10. *The second idea is the "combination" of* two results $x_i : n_i$ and $x_j : n_j$ to one "equivalent" set $c : d$. The results can be combined if

(i) $x_i = n_j$, in which case $c = x_j$ and $d = n_i$,

or (ii) $x_j = n_i$, in which case $c = x_i$ and $d = n_j$.

For instance, 8 : 10 and 6 : 8 can be combined to give 6 : 10; or 8 : 10 and 10 : 12 can be combined to give 8 : 12.

Consider now the use of substitution and combination on a two component system. Given the results $x_1 : n_1$, and $x_2 : n_2$ two courses are possible:

(A) comparing x_1 and n_2 three cases arise:

(i) $x_1 > n_2$. Substitute $n_2 : n_1 n_2/x_1$ for $x_1 : n_1$.
Then the results are $n_2 : n_1 n_2/x_1$ and $x_2 : n_2$.
Combine these to give $x_2 : n_1 n_2/x_1$. Thus the equivalent system results are $x : n$ where

$$x = x_2$$

$$\text{and } n = n_1 n_2/x_1 .$$

(ii) $x_1 < n_2$. Substitute $x_1 x_2/n_2 : x_1$ for $x_2 : n_2$
and combine $x_1 : n_1$ and $x_1 x_2/n_2 : x_1$ to $x_1 x_2/n_2 : n_1$.

then $x = x_1 x_2/n_2$

and $n = n_1$.

(iii) $x_1 = n_2$. No substitution is necessary and the results can immediately be combined to $x_2 : n_1$

Thus $x = x_2$

and $n = n_1$.

(B) comparing x_2 and n_1 (omitting the details) we have:

(i) if $x_2 > n_1$ $x = x_1$ and $n = n_1 n_2/x_2$,
(ii) if $x_2 < n_1$ $x = x_1 x_2/n_1$ and $n = n_2$,
(iii) if $x_2 = n_1$ $x = x_1$ and $n = n_2$.

Out of the two courses (A) and (B) we define the optimum one to the be one that produces the greatest n. Assuming without loss of generality that $n_1 \geqslant n_2$, it is easy to see that course A is optimum. Since $n_1 \geqslant n_2 \geqslant x_2$ course (B) gives $n = n_2$ which is less than the n that (A) gives, irrespective of the size of x_1 .

Summarizing, for a two component system the results are first ordered so that $n_1 \geqslant n_2$, then a substitution is made so that $x_1 = n_2$ or $x_2 = n_1$, then the results are combined, giving

(i) if $x_1 > n_2$ $x = x_2$ and $n = n_1 n_2/x_1$,
(ii) if $x_1 = n_2$ $x = x_2$ and $n = n_1$,
(iii) if $x_1 < n_2$ $x = x_1 x_2/n_2$ and $n = n_1$.

According to the rules of substitution and combination this method produces equivalent system results with the largest possible sample size.

As an example consider the data 4 : 5 and 10 : 14. First we reorder to make $n_1 \geqslant n_2$ giving 10 : 14 and 4 : 5. Comparing x_1 and n_2 we find x_1 is greater than n_2 and so we must substitute to reduce x_1 to n_2. Thus 5 : 14 × 5/10, i.e. 5 : 7, is substituted for 10 : 14. We now have 5 : 7 and 4 : 5. These can be combined to give 4 : 7. Thus $x = 4$ and $n = 7$, and the

approximate lower 0·9 confidence limit for p obtained from tables of exact binomial limits given in Crow *et al.*[10] is 0·279.

As a second example, with the data 30 : 41, 30 : 36 substitution for the second component gives 30 : 41, 25 : 30 and combination gives 25 : 41 so that the lower 0·9 approximate limit is 0·417.

The rationale behind the substitution procedure is that by substituting one is throwing away information, and therefore inferences made from substituted data should be on the conservative side of the exact inferences. For instance, using the exact binomial theory given in section 2, it can be shown that, with only one component, lower confidence limits from substituted data are always smaller than those derived from the original data.

The rationale behind the combination procedure is more dubious. It was suggested by the work of Winterbottom and Verral which is discussed in section 7. The idea is that the component results $x_1 : n_1$, $x_2 : n_2$ where $x_1 = n_2$ are exactly consistent with system results $x_2 : n_1$ where, unknown to the system tester, $n_1 - x_1$ failures were caused by the first component, and $x_1 - x_2$ were caused by the second.

Intuitively one feels that this approximate method should give conservative limits. In all cases investigated so far this has proved to be the case.

The procedure is easily generalized to the case of m components. To begin with the order of reduction is specified. Let us assume the order is $1, 2, \ldots, m$. First $x_1 : n_1$ and $x_2 : n_2$ are reduced to give $x_2^1 : n_2^1$ then $x_2^1 : n_2^1$ and $x_3 : n_3$ are reduced to give $x_3^1 : n_3^1$, and so on. For instance, the results 4 : 5, 9 : 10, 5 : 6, 7 : 8 give, successively,

4 : 5.56, 5 : 6, 7 : 8
3.6 : 6, 7 : 8
3.6 : 6.86.

That is, $x = 3.6$ and $n = 6.86$.

Different orders for reduction can give different final values of x and n. An optimal order is one that gives the largest possible value of n. It is proved in Appendix 1 that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m$ is an optimal order (this has already been shown to be the case for $m = 2$). For instance, with the data given above, starting with the order

9 : 10, 7 : 8, 5 : 6, 4 : 5

one gets

7 : 8.89, 5 : 6, 4 : 5
5 : 7.619, 4 : 5
4 : 7.619.

Thus the proposed method is to first reorder the component data so that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m$, and then to reduce it using the rules of substitution and combination given above.

This method is the same as Lindstrom and Madden's method when the sample sizes are equal ($n_1 = n_2 = \cdots = n_m$). When the sample sizes are not all equal it gives a greater value of n in all cases except when $x_i = n_i$, $i = 1, 2, \ldots, m - 1$, where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m$; and therefore it gives larger lower confidence limits. As with Lindstrom and Madden's method it is simple enough to be executed by hand with the aid of tables of exact binomial limits.

6. A COMPARISON OF COMPETING METHODS

Features considered desirable for approximate methods are:

(1) they should give limits that are reasonably close to the exact limits,
(2) the achieved confidence levels should be greater than the intended levels,
(3) consistent with (2) above, the achieved confidence should be as small as possible,
(4) the limits should be easy to calculate.

Winterbottom[11] gives some comparisons of limits obtained by the following approximate methods:

MMLI	a version of the method of Easterling outlined in section 4
LM	the method of Lindstrom and Madden outlined in section 4
AML	the asymptotic maximum likelihood method outlined in section 4
Mann	the method given by Mann *et al.*[14] and summarised in Winterbottom[11].
Madansky	the method of Madansky[13]
CW	the method of Connor and Wells[15].

In addition to these we shall consider:

MML	another version of the method of Easterling
SR	the method of "successive reduction" proposed in Section 5.

Exact methods that will be referred to are:

UMVE	the exact method based on ordering by the uniformly minimum variance unbiased estimate of p.
LP	the exact method based on the ordering that gives the "largest possible" limits (see Section 3).
Lipow & Riley	the exact limits tabulated in Lipow and Riley[12].

Only systems with 2 components ($m = 2$) will be considered in this section. This is primarily because this is by far the simplest case computationally; the effort required rises enormously as m increases. We believe that many features can be successfully illustrated by this case, however more work needs to be done for higher values of m to validate some of the conclusions.

To begin the comparison we consider the question of which of these methods have feature (2) above, i.e. which methods actually achieve the confidence that they set out to achieve. Let us look first at the confidence levels achieved by one of the exact methods, namely the method used by Lipow and Riley[12] in constructing their tables. An extract from the Lipow and Riley tables is shown in Table 2. For the case $n_1 = n_2 = 10$, Table 2 gives the lower 0·9 confidence limit $L_{0\cdot9}(x_1, x_2)$ for p as a function of x_1 and x_2. For fixed values of p_1 and p_2 the confidence level actually achieved is the sum of the probabilities of all pairs (x_1, x_2) such that

$$L_{0\cdot9}(x_1, x_2) \leqslant p_1 p_2$$

where

$$Pr\{x_1, x_2\} = \binom{n_1}{x_1} {p_1}^{x_1} (1 - p_1)^{n_1 - x_1} \binom{n_2}{x_2} {p_2}^{x_2} (1 - p_2)^{n_2 - x_2}.$$

The confidence level achieved depends on the values of p_1 and p_2 and thus varies over the unit square. Thus it requires a 3 dimensional graph to display the full range of values. To overcome this problem the achieved confidence levels have been calculated for values of p_1 and p_2 that lie on the lines shown in Figure 3. The equation for this family of lines is

$$p_i = 1 - K(1 - \sqrt{p_1 p_2})$$

where i (= 1 or 2) and K ($0 \leqslant K \leqslant 1$) are parameters identifying particular lines in the family. Figure 4 shows the confidence levels achieved by the limits given in Table 2 along the lines determined by $i = 1$ and $K = 1$, 0·8, 0·1 and 0·01. In these figures the abscissa p equals $p_1 p_2$.

Figure 4 shows "cuts" through the surface of achieved confidence over the unit square. It can be seen that the surface comes down and touches the plane of confidence 0·9 at a series of points over the unit square, just as in the 1 dimensional case shown in Figure 1 the curve comes down and touches the line of confidence 0·9. There is at least one such touch point in the plane for each confidence limit shown in Table 2, and this point lies somewhere on the line

$$p_1 p_2 = \text{that limit.}$$

For instance the value 0·5504 is a limit shown in Table 2 (it corresponds to $x_1 = 8$, $x_2 = 10$ and $x_1 = 10$, $x_2 = 8$). The line

$$p_1 p_2 = 0\cdot5504$$

is shown in Figure 5 and the surface of achieved confidence touches the plane of confidence 0·9 at the points $p_1 = 0{\cdot}5504$, $p_2 = 1$ and $p_1 = 1$, $p_2 = 0{\cdot}5504$. Similarly the value 0·4133 is another limit shown in Table 2, and surface touches the plane at the point $p_1 = p_2 = 0{\cdot}6429$ which lies on the line

$$p_1 p_2 = 0{\cdot}4133.$$

Some other touch points are also shown in Figure 5. Although this case gives the impression that all the touch points lie on one of the lines $p_1 = 1$, $p_2 = 1$ or $p_1 = p_2$, this is not true; in general they lie anywhere in the unit square.

Returning to the consideration of approximate methods, if the surface of achieved confidence for a particular method goes down below the 0·9 plane at any point then it does not achieve the desired confidence, i.e. it does not satisfy criterion 2. As an example consider the MMLI method when $n_1 = 10$, $n_2 = 7$. The cut given by $K = 1$ is shown in Figure 6. Clearly the MMLI method does not guarantee to achieve the desired confidence.

To establish that a particular method does not guarantee the desired confidence, all one needs is one example of it not doing so. To establish that a method does achieve the desired confidence one needs to prove the truth of this proposition in all possible cases. As far as we are aware no such proof exists for any of the approximate methods. However, after examination of a number of cases it seems likely that some methods do have this property.

Table 1 of Winterbottom[11] provides examples that show that the methods MMLI, AML Mann and Madansky do not always achieve the desired level of confidence. This follows from the fact that, with the same ordering as one of the exact methods (see Section 3 for a discussion of ordering), there are cases where the approximate limits are greater than the exact limits. These cases are given in Table 3.

The methods for which we do not know of any such examples are MML, LM, CW and SR.

The LM and CW methods will not be considered further; the LM method because, as was pointed out in Section 5, it is dominated by the SR method (dominated in the sense of always giving equal or smaller limits than), and the CW method because the comparisons of Winterbottom[11] show that it is very similar in performance to the LM method.

The MML and SR methods appear to be the two that give the largest limits consistent with feature (2), which means of course that they are best in terms of feature (3). Both methods generate "equivalent" system results, x successes out of n trials, where x and n are not necessarily integers. Therefore to obtain confidence limits from ordinary tables some form of interpolation must be used. The one used to generate the results discussed below is as follows. Let $[y]$ be the integer part of y. Let $L_\alpha(x, n)$ be the α lower confidence limit given by equation (4) (the notation has been altered slightly from $L_\alpha(x)$ to $L_\alpha(x, n)$). Then, if x and n are not integers, the limit is equal to

$$a + (n - [n])\,(b - a)$$

where $$a = (x - [x])\,(L_\alpha([x] + 1, [n]) - L_\alpha([x], [n])) + L_\alpha([x], [n])$$

and $$b = (x - [x])\,(L_\alpha([x] + 1, [n] + 1) - L_\alpha([x], [n] + 1)) + L_\alpha([x], [n] + 1).$$

An alternative form of interpolation would have been to use (5), since $I(s, \alpha, \beta)$ is defined for real values of s, α and β.

To enable comparison with results given by Winterbottom[11] some of the limits produced by the MML and SR methods in the cases $n_1 = 10$, $n_2 = 7$ and $n_1 = 10$, $n_2 = 10$ are given in Table 4. The limits given by the exact LP and UMVE methods are also given.

Easterling[2] pointed out that the MML method runs into difficulties in cases where no failures are observed. In his analysis he used the rule "For the case of no failures the pseudo-sample size was taken to be min $(n_1, n_2)\cdots$". This statement is slightly ambiguous, it could mean:

(a) if $x_1 = n_1$ *and* $x_2 = n_2$, let $\hat{n} = \min(n_1, n_2)$,
or
(b) if $x_1 = n_1$ or $x_2 = n_2$, let $\hat{n} = \min(n_1, n_2)$.

Interpretation (a) would certainly lead to a rule where the achieved confidence was sometimes less than the intended confidence. Taking an extreme example, with $n_1 = 30$, $n_2 = 2$ the results

$x_1 = 29$, $x_2 = 2$ would give $\hat{x} = 29$, $\hat{n} = 30$ and a lower 0·9 confidence limit of 0·876. However the exact limit must be less than or equal to that obtained from 2 successes in 2 trials, viz. 0·316.

For this reason we have used interpretation (b). Thus the results $n_1 = 30$, $n_2 = 2$, $x_1 = 29$ $x_2 = 2$ give $\hat{n} = 2$, $\hat{p} = 0{\cdot}967$, $\hat{x} = 1{\cdot}933$ and lower 0·9 limit of 0·298.

An investigation of the performance of the SR method was carried out. The cases investigated are listed in Table 5. In no case was the achieved confidence level lower than the intended level. As a typical example of the results achieved, Figure 7 shows four "cuts" through the confidence surface for the case $n_1 = 20$, $n_2 = 10$. Figure 8 shows a comparison of three "cuts" for the SR and MML methods when $n_1 = 10$, $n_2 = 7$. For the case $k = 1$ the SR method is better for values of p above 0·5, the two methods are roughly equal for $0{\cdot}17 \leqslant p \leqslant 0{\cdot}5$, and the MML method is better for p less than 0·17. Since in practice one is rarely interested in systems with $p < 0{\cdot}5$ the SR method seems to have the edge on this comparison. In the case $i = 1$, $K = 0{\cdot}01$ the performances are virtually identical. When $i = 2$, $K = 0{\cdot}01$ (in this case $p_2 \simeq 1$. and p varies with p_1 from 0 to 1) the MML method performs poorly. This case highlights the difficulty encountered by the MML method when there are no failures on the component with the smallest sample size.

Figure 9 shows a further comparison of the two methods when $n_1 = n_2 = 10$. The MML method is superior in this case.

Based on a fairly small amount of evidence our tentative conclusions are that the MML method performs better when the sample sizes are equal or nearly so, and the SR method performs better when the sample sizes differ significantly. However more evidence is needed, especially on structures with more than two components.

7. A SEQUENTIAL PROCEDURE FOR SERIES AND PARALLEL SYSTEMS

The discussion in the previous sections has been concerned with the problem of constructing confidence limits for system reliability with data obtained from fixed sample size tests of the components. Clearly this is a difficult problem. The problem can be avoided if the sample sizes are not fixed in advance, but are chosen sequentially throughout the testing of the components. Winterbottom and Verrall[4] give such a scheme.

The scheme for testing the components of a series system is as follows. Test component 1 n times giving x_1 successes and $n_1 - x_1$ failures. Test component 2 $n_2 = x_1$ times giving x_2 successes and $n_2 - x_2$ failures, and so on, with component i being tested $n_i = x_{i-1}$ times giving x_i successes and $n_i - x_i$ failures ($i = 2, 3, \ldots, m$). Winterbottom and Verrall[4] show that x_m is an observation on a binomial random variable with probability p and sample size n. Confidence limits for p can be constructed from x_m and n by the method given in Section 2. Thus the rule used for testing components linked in series is to use the number of successes obtained for component i as the sample size for component $i + 1$.

Winterbottom and Verrall[4] give a similar rule for testing a system whose components are linked in parallel, i.e. a system with reliability

$$p = 1 - \prod_{i=1}^{m} (1 - p).$$

The rule is that the sample size for component i is the number of failures obtained on component $i - 1$.

These two rules can be combined for any system that can be broken down into blocks consisting of units arranged in series or parallel. The results can still be expressed as an observation on a binomial random variable with sample size n and probability p.

Often it is impossible for practical reasons to be able to run sequential tests of this type. In those cases fixed sample size tests must be used and the methods discussed in the previous sections apply. However, in cases where sequential tests can be used, the scheme of Winterbottom and Verrall is efficient in the sense that it minimizes the number of tests required and it provides data from which confidence intervals can be calculated easily by exact methods.

8. A GENERAL SEQUENTIAL PROCEDURE

Winterbottom and Verrall's scheme was generalized by Preston[5] to deal with any coherent system whose structure function is known. This procedure determines the number (n_i) of tests that must be performed on each component, and schedules these tests in such a way that the results can be expressed as an observation on a binomial random variable with sample size n and probability p.

We set out to perform the equivalent of n tests on the whole system. It will be convenient to refer to these as single tests on n identical 'systems'. Thus if x_{ij} (= 1 or 0) is the result of the *jth* test on the *ith* component the vector $(x_{1j}, x_{2j}, \ldots, x_{ij}, \ldots x_{mj})$ will be the result of the tests on 'system' j.

It is not always necessary to carry out n tests on all components, for the following reason. Suppose we have just finished testing component r and that the results of the tests on the components of 'system' j so far obtained are $(x_{1j}, x_{2j}, \ldots, x_{rj})$. If

$$f(x_{1j}, x_{2j}, \ldots, x_{rj}, z_{r+1}, \ldots, z_m) = 1$$

for all values of the vector $(z_{r+1}, \ldots, z_m)$ where $z_i = 0$ or 1, then we know that 'system' j will be assessed as reliable, irrespective of the results of the tests on components $r + 1, \ldots, m$. Thus it is not necessary to test these components for 'system' j. Similarly, if

$$f(x_{1j}, x_{2j}, \ldots, x_{rj}, z_{r+1}, \ldots, z_m) = 0$$

for all $(z_{r+1}, \ldots, z_m)$ 'system' j has failed the system test and its remaining components do not need to be tested. Thus, at stage i of the sequential testing procedure we define three classes of 'systems', those that have passed the test, those that have failed, and those whose outcome is not yet determined. We denote these as follows:

'systems' j, $j \in A_i$, have passed
'systems' j, $j \in B_i$, have failed
'systems' j, $j \in C_i$, need further testing.

The sequential procedure runs as follows. Test component 1 n times with results $x_{11}, x_{12}, \ldots, x_{1n}$. Let A_1 be the set of integers j such that

$$f(x_{1j}, z_2, z_3, \ldots, z_m) = 1$$

for all $(z_2, \ldots, z_m)$; and let B_1 be the set of integers j such that

$$f(x_{1j}, z_2, \ldots, z_m) = 0$$

for all $(z_2, \ldots, z_m)$ (A_1 and B_1 will often be null sets). Let C_1 be the set of integers in the range $1, 2, \ldots, n$, not contained in A_1 or B_1. Let s_1 be the number of elements in C_1.

Test component 2 s_1 times and record the results as x_{2j}, $j \in C_1$. Let D_2 be the set of integers j, $j \in C_1$ such that

$$f(x_{1j}, x_{2j}, z_3, \ldots, z_m) = 1$$

for all $(z_3, \ldots, z_m)$, and let E_2 be the set of integers j, $j \in C_1$ such that

$$f(x_{1j}, x_{2j}, z_3, \ldots, z_m) = 0$$

for all $(z_3, \ldots, z_m)$. Let

$$A_2 = A_1 \cup D_2,$$
$$B_2 = B_1 \cup E_2,$$
$$C_2 = C_1 - D_2 - E_2,$$

and let s_2 be the number of elements in C_2.

Continuing in this manner, the *ith* step is as follows. Test component i s_{i-1} times and record the results x_{ij}, $j \in C_{i-1}$. Let D_i be the set of integers j, $j \in C_{i-1}$, such that

$$f(x_{1j}, \ldots, x_{ij}, z_{i+1}, \ldots, z_m) = 1 \tag{19}$$

for all $(z_{i+1}, \ldots, z_m)$, and let E_i be the set of integers j, $j \in C_{i-1}$, such that

$$f(x_{1j}, \ldots, x_{ij}, z_{i+1}, \ldots, z_m) = 0$$

for all $(z_{i+1}, \ldots, z_m)$. Let

$$A_i = A_{i-1} U D_i,$$
$$B = B_{i-1} U E_i,$$
$$C = C_{i-1} - D_i - E_i,$$

and let s_i be the number of elements in C_i. This procedure is followed for $i = 2, 3, \ldots, m$.

The probability of a 'system' being classified in the set A_m on completion of testing is just the probability of the system passing a system test, i.e. the probability p. Let x be the number of elements in A_m on completion of testing. Then x is an observation on a binomial random variable with sample size n and probability p. The proof of this proposition is given in Appendix 2. It is easy to show that this procedure reduces to the one given by Winterbottom and Verrall for a series or parallel system.

Example: The structure function for a bridge network (see Figure 10) is

$$f(y_1, y_2, y_3, y_4, y_5) = 1 - (1 - y_1y_2)(1 - y_3y_4)(1 - y_1y_4y_5)(1 - y_2y_3y_5).$$

This function is represented in tabular form in Table 6. The brackets indicate that the values of the component indicators to the left determine the value of y irrespective of the values of the component indicators to the right. For instance, from rows 21 to 24, it can be seen that if $y_1 = 0$, $y_2 = 1$, and $y_3 = 0$,

$$f(y_1, y_2, y_3, z_4, z_5) = 0$$

for all (z_4, z_5).

Suppose $n = 10$ and the results for component 1 are 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, i.e.

$$x_{11} = 1,\ x_{12} = 1,\ x_{13} = 1,\ x_{14} = 0 \text{ etc.}$$

The sets A_1 and B_1 are null,

$$C_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

and $s_1 = 10$.

Component 2 is tested 10 times giving 1, 1, 0, 0, 1, 1, 1, 0, 1, 1.
Pairing-off with the results for component 1 we have

j	1	2	3	4	5	6	7	8	9	10
x_{1j}	1	1	1	0	1	1	0	1	1	1
x_{2j}	1	1	0	0	1	1	1	0	1	1

Then $D_2 = \{1, 2, 5, 6, 9, 10\}$
$A_2 = D_2$
E_2 is null
B_2 is null
$C_2 = \{3, 4, 7, 8\}$
and $s_2 = 4$.

Component 3 is tested 4 times giving 0, 0, 1, 1. The results for the systems in C_2 are

j	3	4	7	8
x_{1j}	1	0	0	1
x_{2j}	0	0	1	0
x_{3j}	0	0	1	1

Thus D_3 is null
$A_3 = A_2$
$E_3 = \{4\}$
$B_3 = \{4\}$
$C_3 = \{3, 7, 8\}$
and $s_3 = 3$.

Component 4 is tested three times giving 1, 1, 0. The results for the systems in C_3 are

j	3	7	8
x_{1j}	1	0	1
x_{2j}	0	1	0
x_{3j}	0	1	1
x_{4j}	1	1	0

Thus $D_4 = \{7\}$
$A_4 = \{1, 2, 5, 6, 7, 9, 10\}$
E_4 is null
$B_4 = B_3$
$C_4 = \{3, 8\}$
ands $s_4 = 2$.

Component 5 is tested twice giving 1, 0. The results for the systems in C_4 are

j	3	8
x_{1j}	1	1
x_{2j}	0	0
x_{3j}	0	1
x_{4j}	1	0
x_{5j}	1	0

Thus $D_5 = \{3\}$
$A_5 = \{1, 2, 3, 5, 6, 7, 9, 10\}$
and $x = 8$.

This set of component results, namely

component	no. of trials	no. of successes
1	10	8
2	10	7
3	4	2
4	3	2
5	2	1

gives the equivalent system result of 8 successes out of 10 trials. A 90% confidence interval for p, calculated by the method given in Section 2, is (0·550, 1.).

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APPENDIX 1

Assuming without loss of generality that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m$, in this appendix it is proved that, using the methods of substitution and combination described in Section 5, $1, 2, \ldots, m$ is an optimal order. That is, it gives the largest possible value of n.

The methods of substitution and combination, when applied together and in that order, will be referred to as the method of "reduction". Thus we speak of reducing $x_i: n_i$, $x_j: n_j$ to $x_k: n_k$, and n_k is called the "result" of the reduction.

A general order will be denoted by $i(1), i(2), \ldots, i(m)$ where $(i(1), i(2), \ldots, i(m))$ is a permutation of $(1, 2, \ldots, m)$. The set $(n_{i(1)}, n_{i(2)}, \ldots, n_{i(m)})$ will be referred to as the sample size vector. The reduction process is always carried out from left to right, so that successive sample size vectors will be

$$\mathbf{N}_1 = (n_{i(1)}, n_{i(2)}, \ldots n_{i(m)})$$
$$\mathbf{N}_2 = (n^1{}_2, n_{i(3)}, \ldots, n_{i(m)})$$
$$\mathbf{N}_3 = (n^1{}_3, n_{i(4)}, \ldots, n_{i(m)})$$

where $\mathbf{N}_1$ has m components, $\mathbf{N}_2$ has $m - 1$ components; and so on, and $n^1{}_2$ is the result of reducing $x_{i(1)}: n_{i(1)}$, $x_{i(2)}: n_{i(2)}$ etc.

THEOREM 1 When $m = 2$ an optimal order is 1, 2 and the substitution rules are:

(i) if $x_1 > n_2$ substitute $n_2: n_1 n_2/x_1$ for $x_1: n_1$
(ii) if $x_1 = n_2$ no substitution is required
(iii) if $x_1 < n_2$ substitute $x_1 x_2/n_2: x_1$ for $x_2: n_2$.

Proof: The proof is given in the text of Section 5.

THEOREM 2 The data $x_1: n_1, x_2: n_2$ give an equal or better result than $\bar{x}_1: \bar{n}_1, x_2: n_2$ where

$$\frac{\bar{x}_1}{\bar{n}_1} = \frac{x_1}{n_1} \text{ and } n_2 \leqslant \bar{n}_1 \leqslant n_1.$$

Proof:

(i) If $\bar{x}_1 > n_2$, $\bar{x}_1: \bar{n}_1, x_2: n_2$ give the result $\bar{n} = \bar{n}_1 n_2/x_1$; and $x_1: n_1, x_2: n_2$ give the result $n = n_1 n_2/x_1 > \bar{n}$.

(ii) If $\bar{x}_1 = n_2$, $\bar{x}_1: \bar{n}_1, x_2: n_2$ give $\bar{n} = \bar{n}_1$ and $x_1: n_1, x_2: n_2$ give
$$\begin{aligned} n &= n_1 n_2/x_1 \text{ (since } x_1 > \bar{x}_1 = n_2) \\ &= n_1 \bar{x}_1/x_1 \\ &= \bar{n}_1 \\ &= \bar{n}. \end{aligned}$$

(iii) If $\bar{x}_1 < n_2$, $\bar{x}_1: \bar{n}_1, x_2: n_2$ give $\bar{n} = \bar{n}_1$ and $x_1: n_1, x_2: n_2$ give
$$\begin{aligned} n &\geqslant n_1 n_2/x_1 \\ &> n_1 \bar{x}_1/x_1 \\ &= \bar{n}_1 \\ &= \bar{n}. \end{aligned}$$

THEOREM 3 When $m = 3$ an optimal order is 1, 2, 3.

Proof: An order will be said to dominate another order if it gives at least as great and possibly greater value of n. The possible orders are:

A	1, 2, 3
B	1, 3, 2
C	2, 1, 3
D	2, 3, 1
E	3, 1, 2
F	3, 2, 1

Now, from Theorem 1 the order 1, 2 dominates the order 2, 1. Applying theorem 2 it follows that 1, 2, 3 dominates, 2, 1, 3, i.e. C is dominated by A. Similarly E is dominated by B and F by D. This leaves

A	1, 2, 3
B	1, 3, 2
D	2, 3, 1

An exhaustive analysis of the results obtained from each of these orders is given in Table 7. Examination of every case shows that order A always gives a value of n as great as or greater than orders B and D.

THEOREM 4 For any $m \geqslant 2$ the order $1, 2, \ldots, m$ is an optimal order.

Proof: The proof is by induction. Theorems 1 and 3 have established its truth for $m = 2$ and $m = 3$. Assuming its truth for $m - 1$ components it will be shown to be true for m components, where $m \geqslant 4$.

It is obvious that at least one optimal order must exist. Let $i(1), i(2), \ldots, i(m)$ be such an order. Now consider the first $m - 1$ components $i(1), i(2), \ldots, i(m-1)$. These will be successively reduced to give the result $n^1{}_{m-1}$. However, by the inductive assumption, the order $1, 2, \ldots, i(m)-1, i(m)+1, \ldots, m$ must give at least as good a result. Further, by Theorem 2, this must combine with component $i(m)$ to give at least as good a result for the whole system. That is, the order $1, 2, \ldots, i(m)-1, i(m)+1, \ldots, m, i(m)$ must also be optimal. Now three cases arise.

Case (a) $i(m) > 2$.

Beginning with the optimal order $1, 2, \ldots, i(m)-1, i(m)+1, \ldots, m, i(m)$ the first reduction gives the sample size vector

$$(n^1{}_2, n_3, \ldots, n_{i(m)-1}, n_{i(m)+1}, \ldots, n_m, n_{i(m)})$$

where $n^1{}_2 \geqslant n_3$. By the inductive assumption from here on the order $2, 3, \ldots, m$ must be optimal. But clearly the initial order $1, 2, \ldots, m$ would have given an identical result, so it too must be optimal.

Case (b) $i(m) = 2$.

Beginning with the optimal order $1, 3, 4, \ldots, m, 2$ the first reduction gives sample size vector $(n^1{}_2, n_4, \ldots, n_m, n_2)$. Depending on which of $n^1{}_2$ and n_2 is largest, since there are now $m - 1$ components, the order corresponding to either $(n^1{}_2, n_2, n_3, \ldots, n_m)$ or $(n_2, n^1{}_2, n_3, n_4, \ldots, n_m)$ is optimal. Continuing with the appropriate one of these, the next reduction gives sample size vector $(n^1{}_3, n_4, \ldots, n_m)$. Now compare this with what would have been obtained starting with the initial order $1, 2, \ldots, m$. Then $n^1{}_3$ was obtained by reducing the first three components in the order 1, 3, 2, and by Theorem 3 this must give a result less than or equal to that obtained from the order 1, 2, 3. Repeated application of Theorem 2 then shows that the final result obtained from $1, 2, \ldots, m$ must be at least as good as that obtained from the optimal order $1, 3, 4, \ldots, m, 2$. Thus $1, 2, \ldots, m$ must also be optimal.

Case (c) $i(m) = 1$.

Beginning with the optimal order $2, 3, \ldots, m, 1$ the first reduction gives sample size vector

$$(n^1{}_2, n_4, n_5, \ldots, n_m, n_1).$$

Since there are now $m - 1$ components the order corresponding to

$$(n_1, n^1{}_2, n_4, n_5, \ldots, n_m)$$

must be optimal. The next reduction gives

$$(n^1{}_3, n_4, n_5, \ldots, n_m).$$

Here $n^1{}_3$ was obtained from reducing the first three components in the order 2, 3, 1. By Theorem 3 the order 1, 2, 3 must be at least as good, and repeated application of Theorem 2 shows that the initial order $1, 2, \ldots, m$ must also be optimal.

Thus in every case the order $1, 2, \ldots, m$ is an optimal order. This completes the proof.

APPENDIX 2

Additional Notation

i, j, k, r, t integer indices

F_j binary vector; m elements; general element f_{ij}, $i = 1, 2, \ldots, m$; the vectors F_j, $j = 1, 2, \ldots, 2^m$ are arranged in lexicographic decreasing order i.e.

$$F_1 = (1, 1, \ldots, 1, 1, 1)$$
$$F_2 = (1, 1, \ldots, 1, 1, 0)$$
$$F_3 = (1, 1, \ldots, 1, 0, 1)$$
$$F_4 = (1, 1, \ldots, 1, 0, 0), \ldots$$

$F^r{}_j$ binary vector; r elements; general element f_{ij}, $i = 1, 2, \ldots, r$; $F^r{}_j$ is the vector F_j truncated at the rth place: $F^r{}_j = (f_{1j}, \ldots, f_{rj})$ and in particular $F^m{}_j = F_j$.

H_k binary vector; $b(k)$ elements; defined as follows:

$$H_1 = (1),\ H_2 = (0),\ H_3 = (1, 1),\ H_4 = (1, 0)$$

$$H_5 = (0, 1),\ H_6 = (0, 0),\ H_7 = (1, 1, 1), \ldots;$$

i.e. the number of elements starts at 1 and increases in steps of 1, and for a given number of elements, all the binary vectors are listed in lexicographic decreasing order.

$b(k)$ number of elements in H_k; $b(k)$ is the integer part of $\log_2(k + 1)$.

G the set of indices j such that $f(F_j) = 1$ (recall that $f(\ )$ is the reliability function); i.e.

$$f(F_j) = 1, j \in G;\ f(F_j) = 0,\ j \notin G.$$

G_i, V_i, ρ_i the sets G_i, the vectors V_i, and the scalars ρ_i, $i = 1, 2, \ldots, h$, are defined recursively as follows: let k be the smallest integer such that $f(H_k, z_{b(k)+1}, \ldots, z_m) = 1$ for all $(z_{b(k)+1}, \ldots, z_m)$. (A1)

Define G_1 as the set of indices j such that

$$F_j^{b(k)} = H_k. \tag{A2}$$

Define $\rho_1 = b(k)$ and $V_1 = H_k$. Find the next smallest integer k such that (A1) holds, define G_2 as the set of indices j such that (A2) holds, define $\rho_2 = b(k)$ and $V_2 = H_k$; and so on. As an example consider the 2-out-of-4 voting system given in Sec. 8. From Table 6 we see that the smallest k such that (A1) holds is $k = 3$, and $H_3 = (1, 1)$.

Then $G_1 = [1, 2, 3, 4, 5, 6, 7, 8],\ \rho_1 = 2,\ V_1 = (1, 1).$

The next smallest k such that (A1) holds is $k = 12$, $H_{12} = (0, 1, 0)$.

Then $G_2 = [21, 22, 23, 24],\ \rho_2 = 3,\ V_2 = (0, 1, 0).$

Proceeding in a similar fashion we find that

$$G_4 = [\ 9, 10],\ \rho_4 = 4,\ V_4 = (1, 0, 1, 1)$$
$$G_5 = [11, 12],\ \rho_5 = 4,\ V_5 = (1, 0, 1, 0)$$
$$G_6 = [15, 16],\ \rho_6 = 4,\ V_6 = (1, 0, 0, 0)$$
$$G_7 = [17, 18],\ \rho_7 = 4,\ V_7 = (0, 1, 1, 1), \text{ and so on.}$$

h number of sets G_i.

v_{ki} elements of the vector V_i; $k = 1, 2, \ldots, \rho_i$; $i = 1, 2, \ldots, h$.

S_i set of indices j such that $\rho_j = i$ for $i = 1, 2, \ldots, m$. In the example given in the definition of G_i
S_1 is null, $S_2 = [1]$, $S_3 = [2, 3]$, $S_4 = [4, 5, 6, 7, 8, 9]$, $S_5 = [10, 11, 12]$.

$\psi(p, z)$ $p^z (1 - p)^{1-z}$

Proposition

The probability of a 'system' being classified in the set A_m is the probaility of the system's being good, i.e. the probability R.

Proof

The probability of the component configuration represented by any binary vector F_j is

$$Pr\{F_j\} = \prod_{k=1}^{m} \psi(R_k, f_{kj}).$$

G is the set of all component configurations that make the system good. The system reliability, R, is the sum of the probabilities of all such component configurations, i.e.

$$R = \sum_{j \in G} Pr\{F_j\}.$$

It follows from the definitions that the sets G_i, $i = 1, 2, \ldots, h$, are a mutually exclusive and exhaustive collection of subsets of G, i.e.

$$G = \bigcup_{i=1}^{h} G_i.$$

Thus

$$R = \sum_{i=1}^{h} \sum_{j \in G_i} Pr\{F_j\}. \tag{A3}$$

Further, from the definition of G_i, the vectors F_j, $j \in G_i$, have their first ρ_i elements in common; in fact these first ρ_i elements constitute the vector V_i, i.e.,

$$f_{kj} = v_{ki}, \; k = 1, 2, \ldots, \rho_i \text{ for } j \in G_i.$$

Thus

$$\begin{aligned}
\sum_{j \in G_i} Pr\{F_j\} &= \sum_{j \in G_i} \prod_{k=1}^{m} \psi(R_k, f_{ki}) \\
&= \sum_{j \in G_i} \prod_{k=1}^{\rho_i} \psi(R_k, v_{ki}) \prod_{k=\rho_i+1}^{m} \psi(R_k, f_{kj}) \\
&= \prod_{k=1}^{\rho_i} \psi(R_k, v_{ki}) \sum_{j \in G_i} \prod_{k=\rho_i+1}^{m} \psi(R_k, f_{kj}) \\
&= \prod_{k=1}^{\rho_i} \psi(R_k, v_{ki})
\end{aligned} \tag{A4}$$

The last step follows from the fact that, by the definition of G_i, the vectors $(f_{\rho_i+1, j}, \ldots, f_{mj})$, $j \in G_i$, constitute a complete enumeration of the binary vectors with $m - \rho_i$ elements, which implies

$$\sum_{j \in G_i} \prod_{k=\rho_i+1}^{m} \psi(R_k, f_{kj}) = 1.$$

From (A3) and (A4)

$$R = \sum_{i=1}^{h} \prod_{k=1}^{\rho_i} \psi(R_k, v_{ki})$$

$$= \sum_{i=1}^{h} Pr\{V_i\}. \tag{A5}$$

Now consider stage i of the sequential testing procedure (see Sec. 8) and compare (19) with (A1). A 'system' j will be classified into D_i, *i.f.f.*

$$(x_{1j}, x_{2j}, \ldots, x_{ij}) = V_t = (v_{1t}, v_{2t}, \ldots, v_{\rho_t t}) \tag{A6}$$

for some value of t in the range $1, 2, \ldots, h$. Of course, for equation (A6) to hold, V_t must have the same number of elements as the vector of component results $(x_{1j}, x_{2j}, \ldots, x_{ij})$, i.e. ρ_t must equal i. S_i is the set of indices t such that $\rho_t = i$. In other words, S_i represents the set of vectors of component results that would cause a 'system' to be classified into D_i at stage i. It follows that the probability of a system being classified into D_i is the sum of the probability of these vectors, viz.,

$$\sum_{t \in S_i} Pr\{V_t\}.$$

Further, from the sequential testing procedure it is easily seen that

$$A_m = \bigcup_{i=1}^{m} D_i.$$

Thus the probability that a 'system' is in A_m on completion of testing is the sum of the probabilities of being classified into D_i over all values of i (i.e. over all stages), viz.,

$$Pr\{\text{'system' in } A_m\} = \sum_{i=1}^{m} \sum_{t \in S_i} Pr\{V_t\}. \tag{A7}$$

Now the sets S_i are mutually exclusive, and

$$\bigcup_{i=1}^{m} S_i = [1, 2, \ldots, h],$$

so that

$$\sum_{i=1}^{m} \sum_{t \in S_i} Pr\{V_t\} = \sum_{i=1}^{h} Pr\{V_i\}. \tag{A8}$$

From (A5), (A7) and (A8), we have

$$Pr\{\text{'system' in } A_m\} = R. \qquad \text{Q.E.D.}$$

TABLE 1

Exact and approximate lower 90% confidence limits for p when n = 10.

x	$L_{0\cdot9}(x)$	$A_{0\cdot9}(x)$
0	0·	0·
1	0·0105	0·
2	0·0545	0·038
3	0·1158	0·114
4	0·1876	0·201
5	0·2673	0·297
6	0·3542	0·401
7	0·4483	0·514
8	0·5504	0·638
9	0·6632	0·778
10	0·7943	1·000

TABLE 2

"Exact" lower 0·9 confidence limits for p when $n_1 = n_2 = 10$.

x_1	x_2	$L_{0.9}(x_1, x_2)$
10	10	0·7943
9	10	0·6631
10	9	0·6631
9	9	0·6071
8	10	0·5504
10	8	0·5504
8	9	0·4971
9	8	0·4971
7	10	0·4483
10	7	0·4483
8	8	0·4457
7	9	0·4133
9	7	0·4133
6	10	0·3542
10	6	0·3542
7	8	0·3542
8	7	0·3542
6	9	0·3439
9	6	0·3439
7	7	0·2673
6	8	0·2673
8	6	0·2673

TABLE 3

Cases where the approximate lower 0·9 confidence limits are greater than the exact limits.

n_1	n_2	x_1	x_2	Ordering*	Exact Limit	Approximate Method	Approximate Limit
10	7	9	5	LP	0·388	MMLI	0·401
10	7	10	5	LP	0·404	AML	0·495
10	10	9	9	UMVE	0·607	Mann	0·615
10	7	10	5	LP	0·404	Madansky	0·475

* This column gives the ordering used in the determination of the exact limits (see Section 3 for a discussion of ordering).

TABLE 4

Lower 0·9 confidence given by the LP and UMVE exact and MML and SR approximate methods.

$n_1 = 10 \quad n_2 = 7$

x_1	x_2	LP	UMVE	MML	SR
10	5	0·404	0·404	0·404	0·404
7	7	0·448	0·404	0·391	0·448
8	6	0·447	0·404	0·428	0·416
9	5	0·388	0·388	0·365	0·358
7	6	0·385	0·354	0·354	0·354
6	7	0·354	0·354	0·304	0·354
10	4	0·279	0·279	0·279	0·279

$n_1 = 10 \quad n_2 = 10$

x_1	x_2	LP and UMVE	MML	SR
10	10	0·794	0·794	0·794
9	10	0·663	0·663	0·663
9	9	0·607	0·573	0·562
8	10	0·550	0·550	0·550
8	9	0·497	0·481	0·469
7	10	0·448	0·448	0·448
8	8	0·445	0·408	0·392
6	10	0·354	0·354	0·354

TABLE 5

Cases for which the achieved confidence levels of the SR level were determined.

n_1	n_2	i*	K
10	7	1	0·01
10	7	1	0·1
10	7	1	0·5
10	7	1	1·0
10	7	2	0·01
10	10	1	0·01
10	10	1	0·1
10	10	1	0·5
10	10	1	1·0
20	10	1	0·01
20	10	1	0·1
20	10	1	0·5
20	10	1	1·0
20	10	2	0·01
20	10	2	0·1

* i and K determine the "cut" (see Figure 3).

TABLE 6

Structure function for a bridge network.

row	y_1	y_2	y_3	y_4	y_5	y
1	1	1	1	1	1	1
2	1	1	1	1	0	1
3	1	1	1	0	1	1
4	1	1	1	0	0	1
5	1	1	0	1	1	1
6	1	1	0	1	0	1
7	1	1	0	0	1	1
8	1	1	0	0	0	1
9	1	0	1	1	1	1
10	1	0	1	1	0	1
11	1	0	1	0	1	0
12	1	0	1	0	0	0
13	1	0	0	1	1	1
14	1	0	0	1	0	0
15	1	0	0	0	1	0
16	1	0	0	0	0	0
17	0	1	1	1	1	1
18	0	1	1	1	0	1
19	0	1	1	0	1	1
20	0	1	1	0	0	0
21	0	1	0	1	1	0
22	0	1	0	1	0	0
23	0	1	0	0	1	0
24	0	1	0	0	0	0
25	0	0	1	1	1	1
26	0	0	1	1	0	1
27	0	0	1	0	1	0
28	0	0	1	0	0	0
29	0	0	0	1	1	0
30	0	0	0	1	0	0
31	0	0	0	0	1	0
32	0	0	0	0	0	0

TABLE 7

Effect of the orders A, B, D when m = 3 (see the Appendix for details).

Condition	Order	First Reduction	Second Reduction (= n)
$x_1 \geqslant n_2$	A	n_1n_2/x_1	$n_1n_2n_3/x_1x_2$
$x_2 \geqslant n_3$	B	n_1n_3/x_1	n_1n_3/x_1
$n_1n_3/x_1 \geqslant n_2$	D	n_2n_3/x_2	$n_1n_2n_3/x_1x_2$
$x_1 \geqslant n_2$	A	n_1n_2/x_1	$n_1n_2n_3/x_1x_2$
$x_2 \geqslant n_3$	B	n_1n_3/x_1	n_2
$x_2 \leqslant n_1n_3/x_1 < n_2$	D	n_2n_3/x_2	$n_1n_2n_3/x_1x_2$
$x_1 \geqslant n_2$	A	n_1n_2/x_1	$n_1n_2n_3/x_1x_2$
$x_2 \geqslant n_3$	B	n_1n_3/x_1	$n_1n_2n_3/x_1x_2$
$n_1n_3/x_1 < x_2$	D	n_2n_3/x_2	$n_1n_2n_3/x_1x_2$
$x_1 \geqslant n_2$	A	n_1n_2/x_1	n_1n_2/x_1
$x_2 < n_3$	B	n_1n_3/x_1	n_1n_3/x_1
$n_1n_3/x_1 \geqslant n_2$	D	n_2	n_1n_2/x_1
$x_1 \geqslant n_2$	A	n_1n_2/x_1	n_1n_2/x_1
$x_2 < n_3$	B	n_1n_3/x_1	n_2
$n_1n_3/x_1 < n_2$	D	n_2	n_1n_2/x_1
$x_1 < n_2$	A	n_1	$n_1n_2n_3/x_1x_2$
$x_1x_2/n_2 \geqslant n_3$	B	n_1n_3/x_1	n_1n_3/x_1
$n_1n_3/x_1 \geqslant n_2$	D	n_2n_3/x_2	$n_1n_2n_3/x_1x_2$
$x_1 < n_2$	A	n_1	$n_1n_2n_3/x_1x_2$
$x_1x_2/n_2 \geqslant n_3$	B	n_1n_3/x_1	n_2
$x_2 \leqslant n_1n_3/x_1 < n_2$	D	n_2n_3/x_2	$n_1n_2n_3/x_1x_2$
$x_1 < n_2$	A	n_1	$n_1n_2n_3/x_1x_2$
$x_1x_2/n_2 \geqslant n_3$	B	n_1n_3/x_1	$n_1n_2n_3/x_1x_2$
$n_1n_3/x_1 < x_2$	D	n_2n_3/x_2	$n_1n_2n_3/x_1x_2$
$x_1 < n_2$	A	n_1	n_1
$x_1x_2/n_2 < n_3$	B	$\leqslant n_1$	n_1
	D	$\leqslant n_1$	n_1

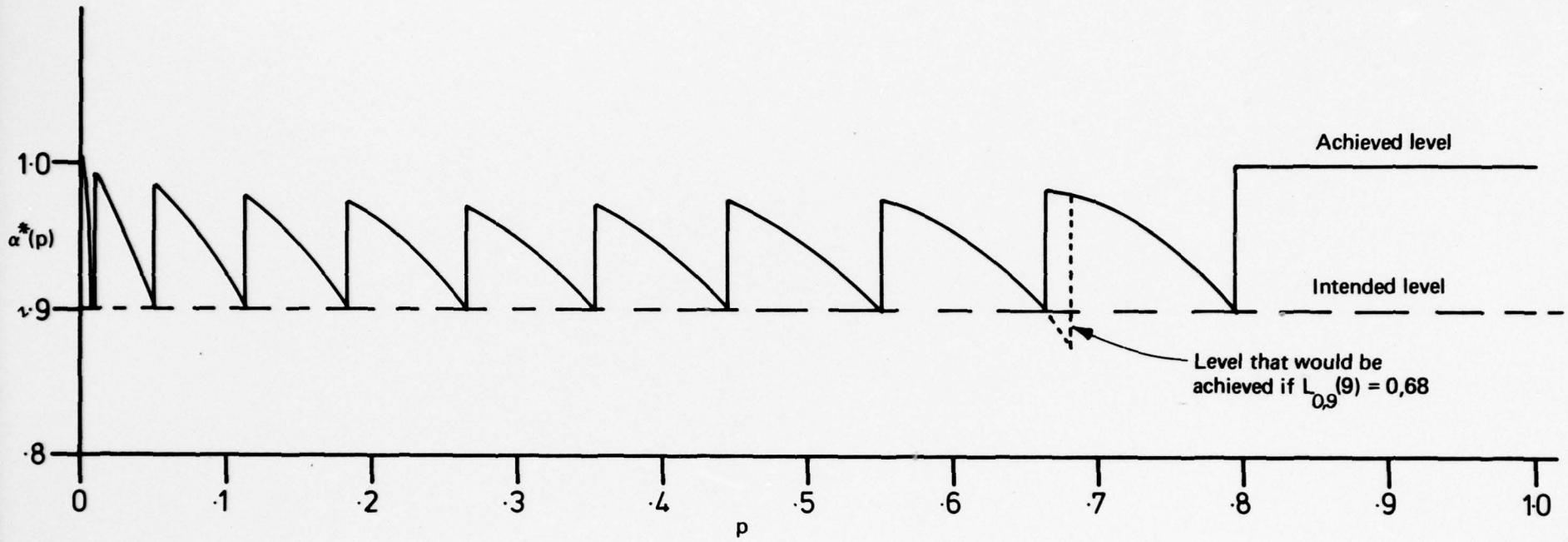

FIG. 1 CONFIDENCE LEVEL ACHIEVED BY EXACT LIMIT WHEN n = 10, α = 0,9

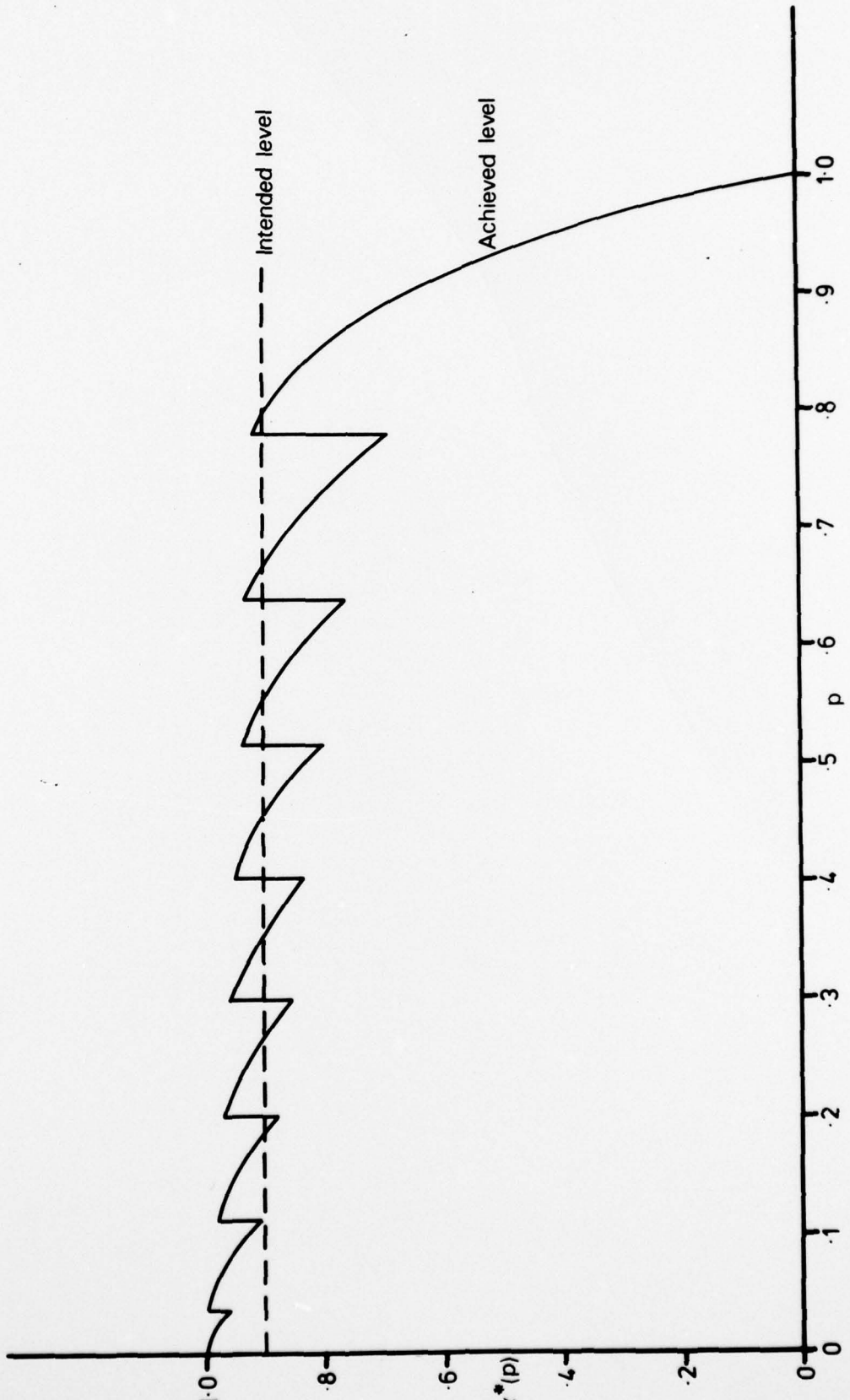

FIG. 2 CONFIDENCE LEVEL ACHIEVED BY APPROXIMATE LIMIT WHEN $n = 10, \alpha = 0{,}9$

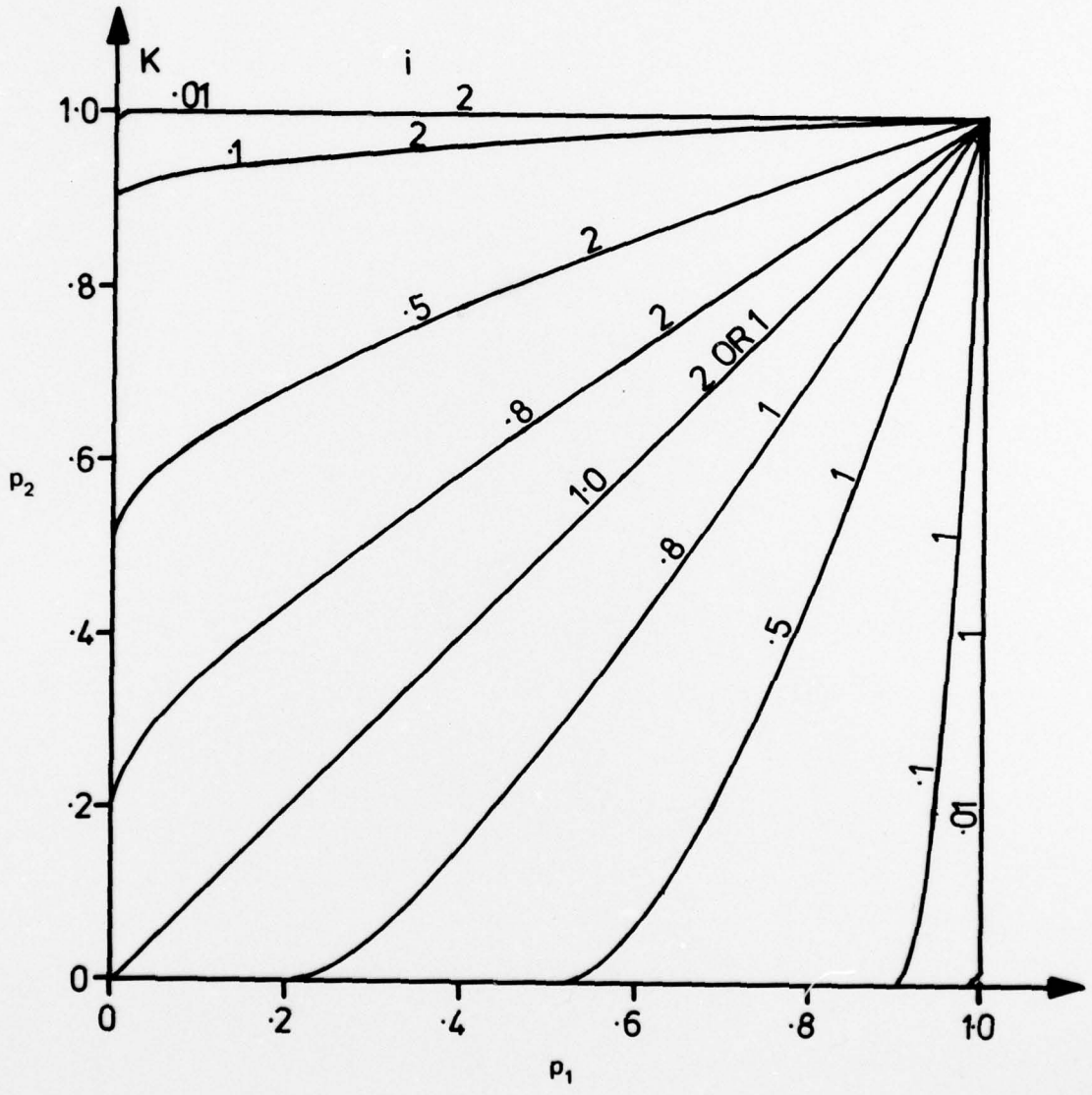

FIG. 3 THE FAMILY OF LINES GIVEN BY $p_i = 1-K(1-\sqrt{p_1\ p_2})$; i = 1, 2; $0 \leq K \leq 1$

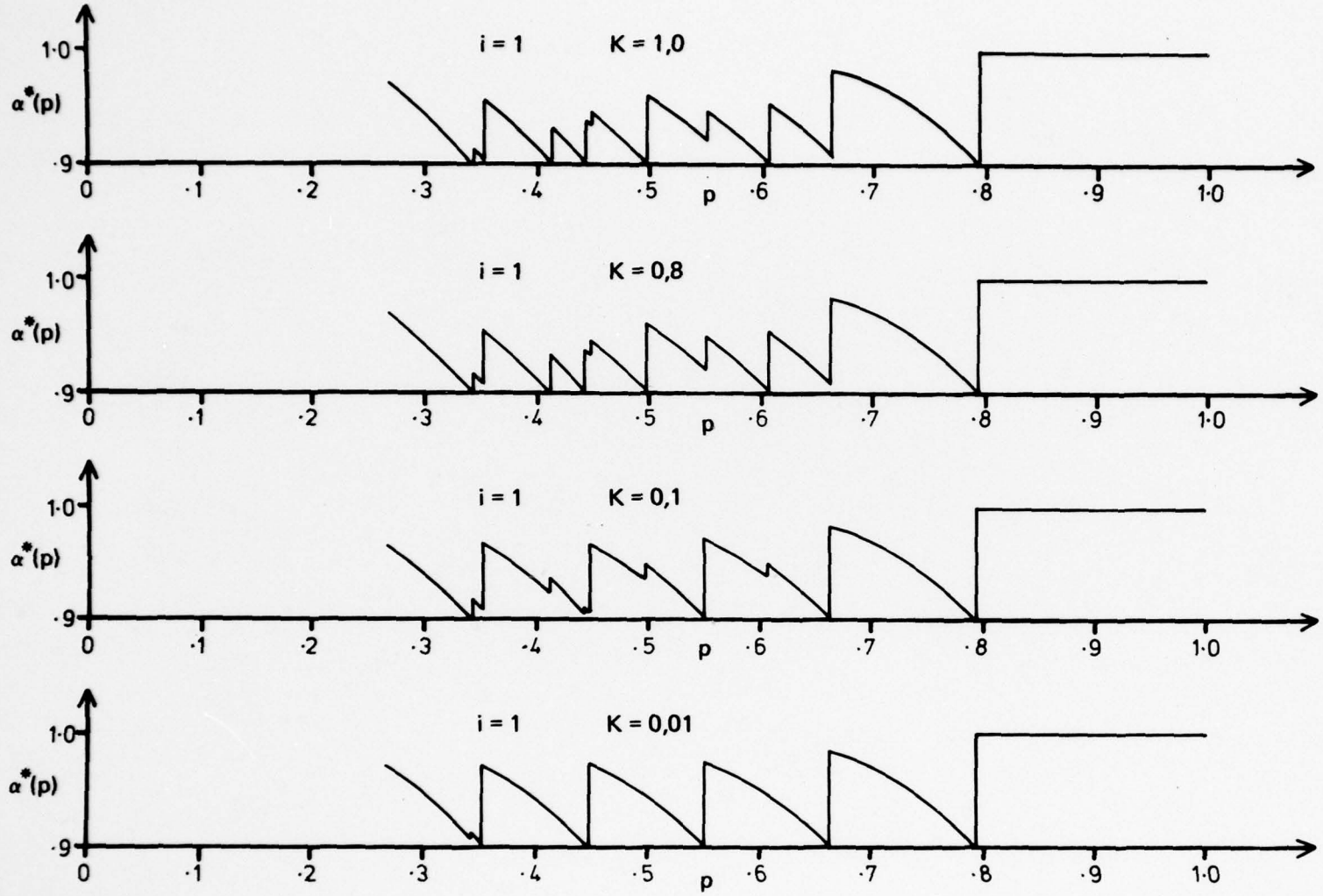

FIG. 4 CONFIDENCE LEVELS ACHIEVED BY EXACT LIMITS WHEN $n_1 = n_2 = 10$

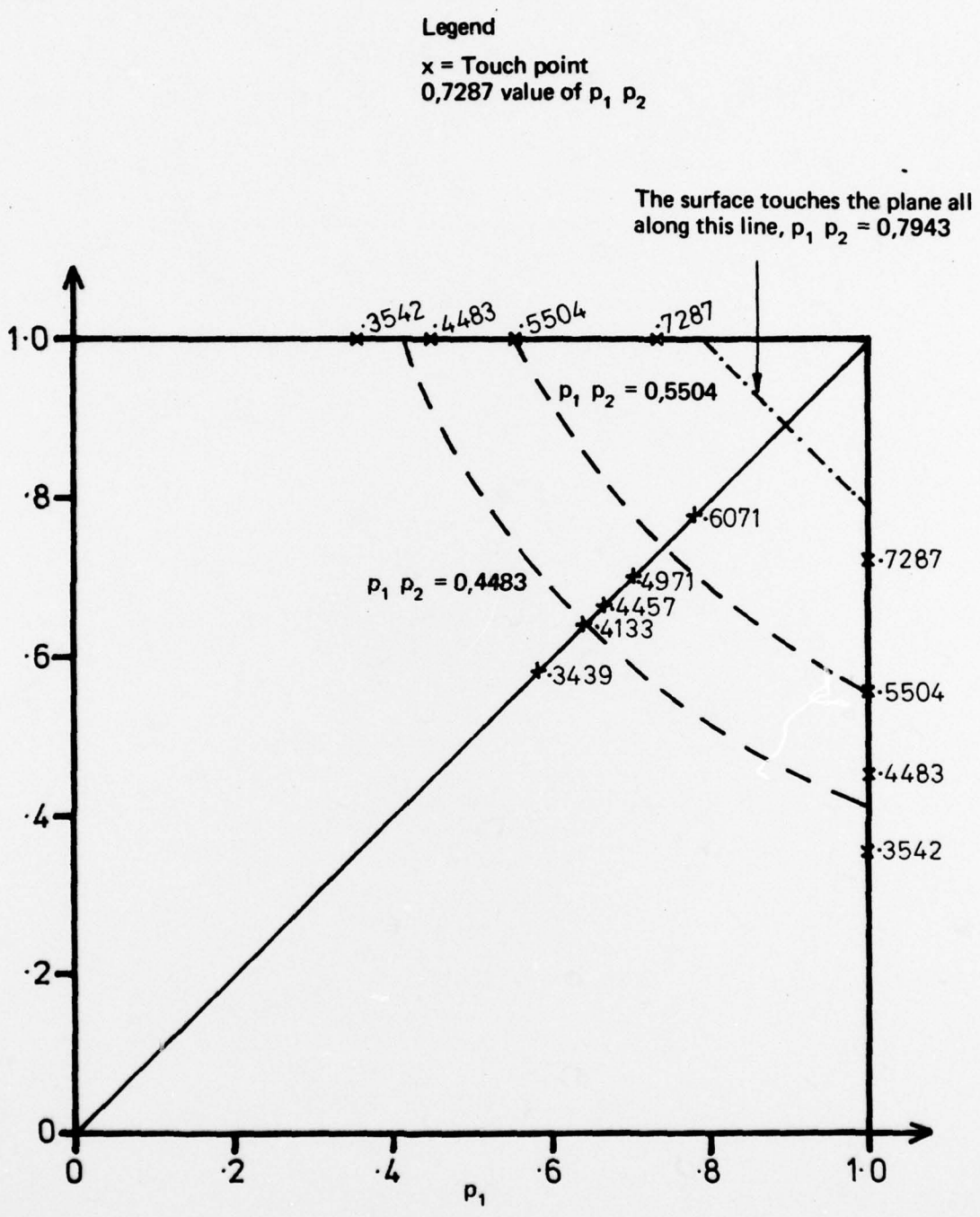

FIG. 5 POINTS AT WHICH THE SURFACE OF ACHIEVED CONFIDENCE OF THE EXACT LIMITS OF LIPOW AND RILEY[12] TOUCHES THE PLANE OF CONFIDENCE 0,9.

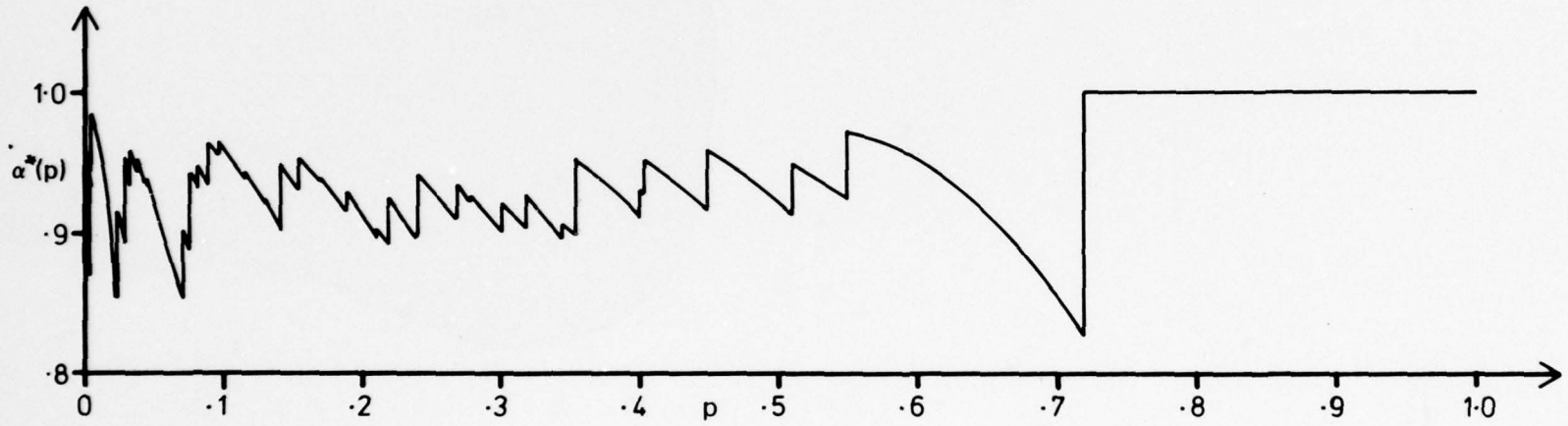

FIG. 6 ACHIEVED CONFIDENCE FOR THE MMLI METHOD ALONG THE CUT K = 1 WHEN $n_1 = 10$, $n_2 = 7$

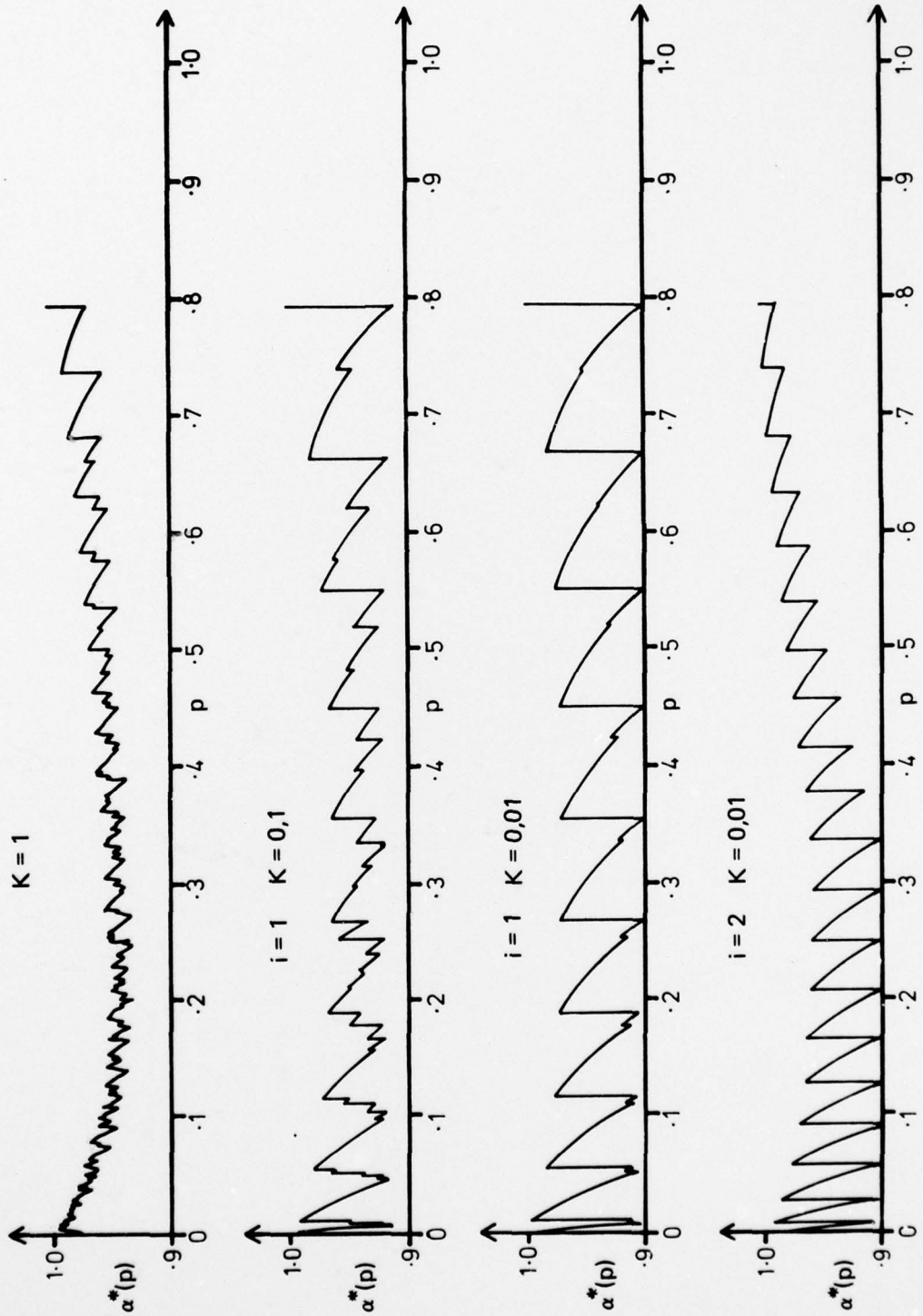

FIG. 7 CONFIDENCE LEVELS ACHIEVED BY THE SR METHOD WHEN $n_1 = 20$, $n_2 = 10$

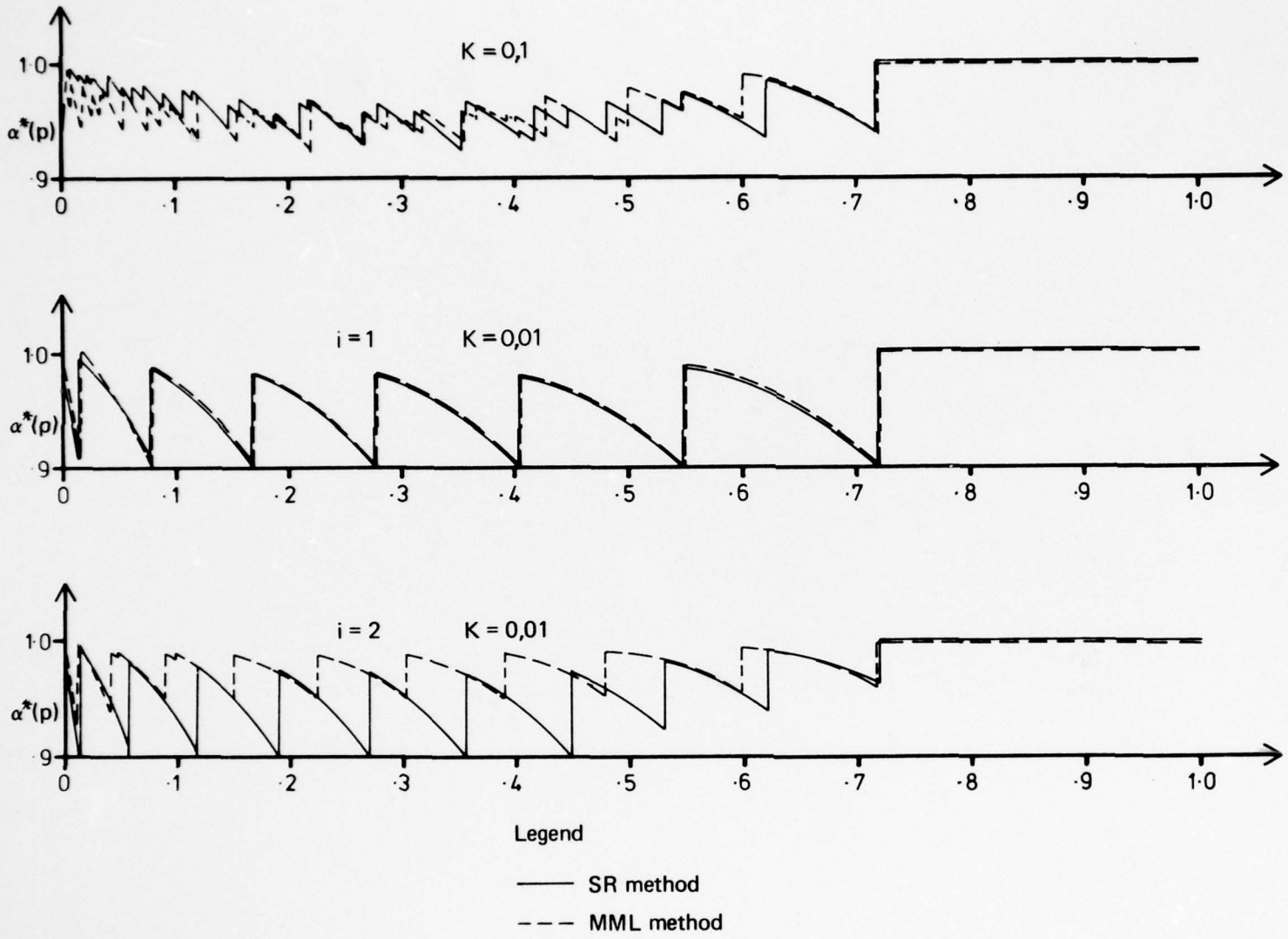

FIG. 8 COMPARISON OF ACHIEVED CONFIDENCE LEVELS OF THE MML AND SR METHODS WHEN $n_1 = 10$, $n_2 = 7$

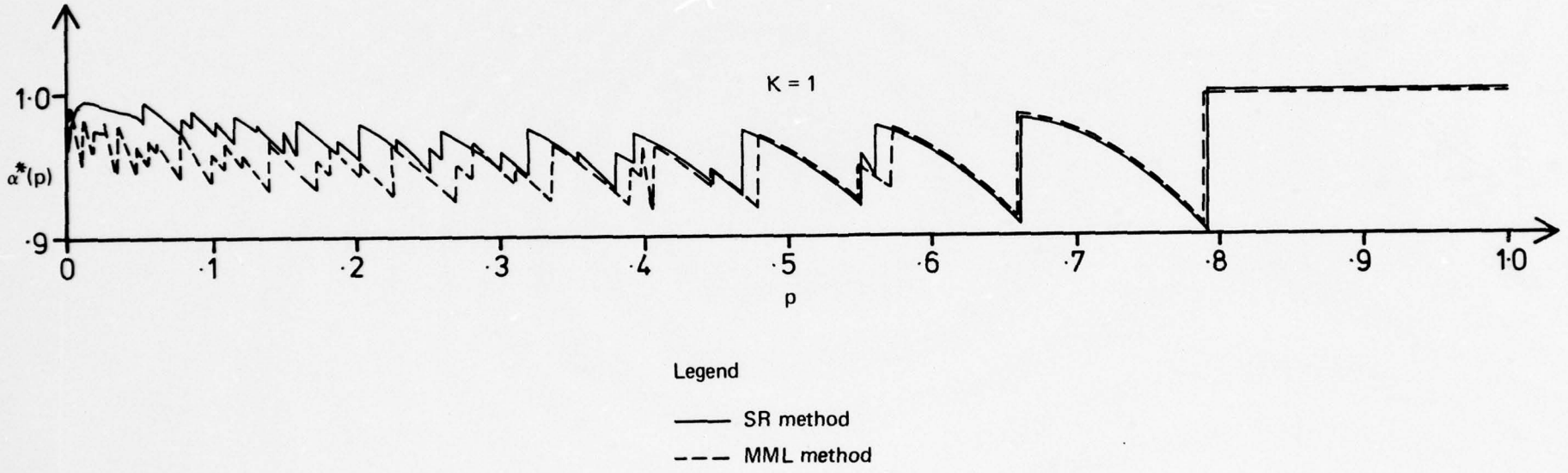

FIG. 9 COMPARISON OF ACHIEVED CONFIDENCE LEVELS OF THE MML AND SR METHODS WHEN $n_1 = n_2 = 10$

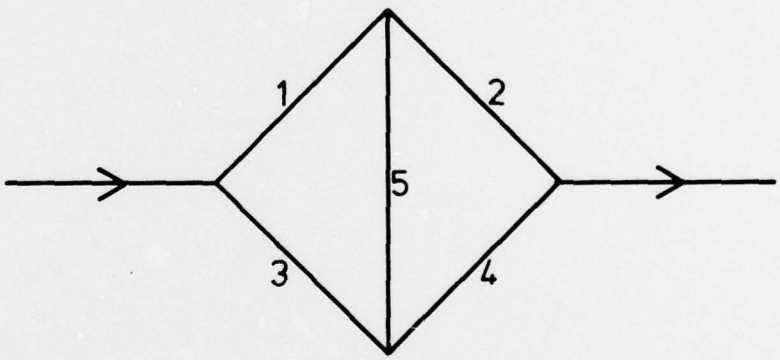

FIG. 10 A BRIDGE NETWORK

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